Trend-Cycle Decomposition Allowing for Multiple Smooth Structural Changes in the Trend of US Real GDP

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Abstract

A key feature of Flexible Fourier Form (FFF) is that the essential characteristics of multiple structural breaks can be captured using a small number of low frequency components from a Fourier approximation. We introduce a variant of the FFF into the trend function of US real GDP in order to allow for gradual effects of unknown numbers of structural breaks occurring at unknown dates. We find that the hypothesis of no breaks can be rejected, and the Fourier components are significant. Our new cycle matches the NBER chronology very well, especially for the Great Recession of 2009.

Keywords: Trend-Cycle Decomposition; Flexible Fourier Form; Smooth Trend Breaks

JEL Classification: E32, E37, C32

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We wish to thank Ting Qin for providing research assistance.

1. Introduction

The appropriate way to decompose real U.S. GDP into its trend and cyclical components has received a substantial amount of attention in the macroeconomics literature. **?** define the trend as the limiting forecast as the horizon goes to infinity, adjusted for the mean rate of growth. As such, trend growth is a random walk plus drift and the cycle is the difference between the actual value of the series and its estimated trend. An identifying restriction in the Beveridge and Nelson (BN) decomposition is that the correlation coefficient between innovations in the trend and the cycle is -1. As is well-known, for the real U.S. GDP series, this restriction produces a stochastic trend that is quite volatile and a cycle that is small.

The unobserved components (UC) approach of **?** decomposes GDP such that the trend is a random walk plus drift and the cycle is a stationary AR(2) process. **?** extend the UC model by allowing the cycle to depend on a cosine wave, and **?** allow the cycle to be a Markov switching process. Unlike the BN decomposition, the UC trend is smooth and the UC cycle is large and highly persistent. This difference is mainly due to the fact that the UC model imposes the restriction that innovations in the trend and cycle are uncorrelated. In a very important paper, **?** (hereafter MNZ) introduce an unrestricted trend-cycle correlation into the UC model and find that it yields the same type of decomposition as the BN method.

Our point of departure from this literature is to note that a misspecified specification of the trend means that the cycle is misspecified as well. The extensions of the BN and UC models have focused on the cycle with relatively little effort going into the specification of the trend. A notable exception is **?** (hereafter PW). PW provide evidence that it is important to account for a structural break in the trend of real U.S. GDP occurring in 1973Q1. After allowing for a break in the drift term, the estimated cycle corresponds very closely to the NBER chronology. Their point is that previous studies misspecified the trend by ignoring the productivity reduction resulting from the oil price shock of 1973. A similar argument is made by **?** for the Canadian economy.

However, PW use a single dummy variable to represent the break. The implicit assumptions are (i) the break date is known to be 1973Q1, or the break is exogenous; (ii) the break has abrupt or instantaneous effect on the trend, and (iii) the number of breaks is only one. In our view all three assumptions are restrictive. First, assumption (i) seems *ad hoc* given the fact the OPEC increased the oil price by 5.7% on April 1, 11.9% on June 1, 17% on October 16, and declared a complete export embargo on October 20. It is unclear which of these dates is the true break date. For assumption (ii), even if the price jumps are best modeled as being sharp, the effects are likely to be gradual as it took time for the price increases to manifest themselves in output reductions. Finally, the assumption of a single break may also be suspect in that a number of studies suggest that the reduction in growth trend actually began sometime in the late 1960s or very early 1970s. For instance, see **?** and the Symposium on the Slowdown in Productivity Growth in the *Journal of Economic Perspectives* of 1988. Moreover, **?** find that there was a gradual resumption of productivity growth in the 1990s that is suggestive of another change in the growth rate of the trend. More recently, the effects of the financial crisis on GDP are certainly indicative of another potentially sharp change in the trend of GDP. The point is that a model allowing for an unknown number of possibly smooth changes in the trend is likely to be superior to a model that contains only a single sharp break.

In this paper, we develop a procedure that allows us to incorporate a smoothly evolving trend in the UC framework. Specifically, we introduce a variant of the Flexible Fourier Form (FFF) of **?** into the trend function. It has been well demonstrated by **?**, **?** and **?** that the essential characteristics of one or more potential trend shifts can be captured using a small number of low frequency components from a Fourier series approximation. In contrast to a dummy variable approach, one desirable feature of the FFF is that we do not need to assume that the break dates or the number of breaks are known *a priori*. A second advantage of the FFF is that it allows for gradual structural change. Other nonlinear models such as the Logistic Smooth Transition Autoregression (LSTAR) can capture smooth breaks as well. But like the dummy-variable approach, the LSTAR requires that the number of breaks be known. Finally, estimating the FFF is much easier than the LSTAR model because the FFF model is linear in the parameters for any given frequency. Below, we follow the convention in the literature and refer to a series with a gradually evolving trend as one containing a number of smooth structural breaks.

The rest of this paper is organized as follows. Section 2 provides a description of the FFF and how it can be used to approximate a nonlinear trend. Forces such as changing population dynamics, exogenous changes in real oil prices, a nonconstant rate of technical change, and events such as the Great Recession can all induce changes in the trend of real U.S. GDP. We show that the FFF can approximate both smooth and sharp breaks so that the functional form of the structural change need not be pre-specified. Section 3 presents our FFF trend-cycle decomposition. Sections 4 and 5 discuss estimation results and compares the decomposition from our model to those from other models. Section 6 concludes.

2. Approximating Structural Breaks with the Fourier Form

Consider a time-series with the intercept term $\alpha(t)$, which is time-varying because of smooth structural change. When the functional form of the change is unknown, it is typical to use a dummy variable approach so as to capture any structural breaks in the data. The methodology can be problematic because the nature of the breaks (smooth or sharp), the number of breaks, and the location of the breaks are typically unknown to the researcher. Instead of the dummy variable approach, we consider a Fourier series approximation for $\alpha(t)$ given as:

$$\alpha(t) \approx \mu + \sum_{k=1}^{n} a_k \sin(2\pi kt/T) + \sum_{k=1}^{n} b_k \cos(2\pi kt/T), \quad (n \leq T/2) \tag{1}$$

where n represents the number of frequencies, k represents the index for frequencies, and T is the number of observations. Note that the traditional model of a constant intercept is nested within alphat. If the value of $\alpha(t)$ is actually constant over time, we should not be able to reject the null hypothesis that all values of a_k and b_k equal 0.

In principle, the Fourier series is capable of approximating absolutely integrable functions to any desired degree of accuracy. Beginning with $n = 1$, it is always possible to improve the approximation by using additional frequencies. When $n = T/2$ is reached, the fit will be perfect. However, in practice it is necessary to restrict the number of frequencies in the estimating equation. As such, instead of the selecting break dates and the form of the structural change, the specification problem is transformed into one of incorporating the appropriate frequency components into the model.

As shown in **?**, **?** and **?**, both sharp and smooth breaks can often be captured using a small number of low frequency components from the Fourier approximation[1]. This is so since breaks or gradual trend shifts, shift the spectral density function towards zero. As such, alphat using a small number of low frequency components can approximate many forms of sharp and/or smooth structural change. Since the aim of using the FFF is to approximate the form of the structural change, the FFF can be thought of as a semiparametric model. Remarkably, the FFF provides a global, rather than a local, approximation. Other approximations, such as Taylor series, are valid only at a particular point in the sample space.

For the purpose of decomposing GDP into a trend and cycle, the key insight is that the FFF can also approximate changes in the slope of a stochastic trend. Consider a random walk with drift

$$y_t = \alpha(t) + y_{t-1} + e_t \tag{2}$$

[1]For instance, Figure 1 in **?** applies the Fourier approximation to one sharp and two sharp (U-shaped) breaks in intercept, a continuous trend with break, and a discontinuous trend with break.

where $\alpha(t)$ can still be approximated by alphat, but now $\alpha(t)$ is interpreted as the drift term. The time-varying drift term results in the time-varying slope of the trend.

Panels A and B of Figure 1 illustrate the ability of the FFF to mimic the behavior of smooth changes in the trend of series. The solid line in Panel A shows the time path of a series with two simulated LSTAR declines in the intercept. Consider:

$$y_t = d_0 + d_1/[1 + \exp(\gamma_1(t - \lambda_1 T))] + d_2/[1 + \exp(\gamma_2(t - \lambda_2 T))] + \delta t \tag{3}$$

where $d_0 = 4, d_1 = -3, d_2 = -1, \gamma_1 = \gamma_2 = -0.1, \lambda_1 = 1/3, \lambda_2 = 3/4$, $\delta = 0.03$, and $T = 267$ (representing the 267 observations in our data set). Note that the first decline is centered around $t = 89$ and the second decline is centered around $t = 200$. The dotted line in the Panel A represents results of regressing y_t on a constant, a time trend, and the first frequency component of a Fourier expansion (i.e., $\sin(2\pi t/267)$ and $\cos(2\pi t/267)$). The dashed line in Panel A shows the effects of setting $n = 2$ and using the first two frequency components of alphat. Hence, the two fitted values are given by:

$$\hat{y}_t = \hat{\mu} + \hat{\alpha}_1 \sin(2\pi t/T) + \hat{\beta}_1 \cos(2\pi t/T) + \hat{\delta} t \tag{4}$$

$$\hat{y}_t = \hat{\mu} + \hat{\alpha}_1 \sin(2\pi t/T) + \hat{\beta}_1 \cos(2\pi t/T) + \hat{\alpha}_2 \sin(4\pi t/T) + \hat{\beta}_2 \cos(4\pi t/T) + \hat{\delta} t \tag{5}$$

The two smooth breaks in Panel A are reinforcing because d_1 and d_2 have the same signs. By contrast, Panel B considers two offsetting LSTAR changes. The data generating process is still sim1, but now $d_0 = 4, d_1 = -1, d_2 = 3, \gamma_1 = -0.2, \gamma_2 = -0.1, \lambda_1 = 1/5, \lambda_2 = 3/4$, $\delta = 0.03$, and $T = 267$. There are three differences. First, the two breaks are offsetting since d_1 and d_2 have the opposite signs. Second, the first break occurs earlier than Panel A. Third, the smoothness of the first break γ_1 differs from Panel A. Again, the fitted values using our Fourier approximation with one and two frequency components are shown by the dotted and dashed lines in the figure (denoted by 1-Frequency and 2-Frequencies, respectively).

Even though the Fourier series is especially suitable for smooth breaks, the series shown in Panels C and D of Figure 1 each contain two sharp breaks:

$$y_t = 0.2t(t < T/3) + (0.1T/3 + 0.1t)(T/3 \leq t < 0.75T) + (28.9)(t \geq 3T/4) \qquad (6)$$

$$y_t = 0.2t(t < T/3) + (0.1T/3 + 0.1t)(T/3 \leq t < 0.75T) + (-31 + 0.3t)(t \geq 3T/4) \qquad (7)$$

The simulated functions in sim3 and sim4 both exhibit sharp declines in the trend growth at $T/3$, sim3 becomes flat at $3T/4$, whereas the trend growth surges at $3T/4$ in sim4. Nevertheless, the dashed lines in the two panels indicate good fit.

Of course, it is possible to use the FFF when a series contains the type of smooth breaks of Panels A and B combined with the sharp breaks of Panels C and D. The nonlinearities depicted in the simulated data are far more pronounced than those actually present in U.S. GDP data. Instead, the solid lines in Panels E and F of Figure 1 are the quarterly values of the logarithms of real U.S. potential and actual GDP, respectively. It is clear that the dashed lines, representing the fitted values of the FFF using $n = 1$, and $n = 2$, are extremely close to the actual values.

Despite the simplicity of the FFF, the first four panels of Figure 1 suggest that a Fourier series approximation using the single frequency $n = 1$ can often mimic a nonlinear trend with a fairly high degree of accuracy. Adding an additional frequency component seems most useful if the structural change has pronounced humps or is especially sharp. Also, the second frequency improves the fit at the starting and ending values of the sample. This is important because much of our interest is on the recent history of GDP in the aftermath of the financial crisis. In Section 4 we will revisit this issue.

In all six panels of Figure 1, the fitted values were obtained using the same functional forms as those in fff1 and fff2. This is a clear advantage of the FFF relative to some other popular methods of approximating a nonlinear trend. Unlike the use of a polynomial function in time using t, t^2,and t^3, the trigonometric functions used by the Fourier form are always bounded. In contrast to incorporating sharp (or smooth) dummy variables into the trend function of the UC model, the FFF does not require that the form of the nonlinearity, the number of breaks, or the break dates be known. Even if the number of breaks is known, estimating an LSTAR or ESTAR function is much harder than the FFF because the STAR function is nonlinear in the parameters (for instance, γ and λ in the LSTAR function are notorious for being hard to identify). Perhaps most importantly, unlike functional forms in which the break dates need to be estimated, the FFF form does *not* entail a nuisance parameter problem in that the null hypothesis of linearity can be expressed as $H_0 : \alpha_1 = \alpha_2 = ... = \alpha_n = \beta_1 = \beta_2 = ... = \beta_n = 0$.

Note that estimating the value of n entails the estimation of the α_k and β_k but not any of the values of k. Intuitively, the problem is analogous to that of selecting n in fitting $\alpha(t) = \alpha_0 + \alpha_1 t + ... + \alpha_n t^n$. Under the null of linearity, the null hypothesis $H_0 : \alpha_1 = \alpha_2 = ... = \alpha_n = 0$ can be tested using a standard χ^2 test. As shown in **?**, the test follows a χ^2 distribution. Hypothesis testing is facilitated since all of the regressors in the FFF (i.e., the $\sin(2\pi kt/T)$ and $\cos(2\pi kt/T)$ functions) are orthogonal to each other. In section 5 we will compare the finite-sample performance of this test to the standard test for linearity.

3. The Model

Our new trend-cycle decomposition employs the same set-up as **?** except that the FFF (trigonometric terms) are included in the trend:

$$y_t = \tau_t + c_t \tag{8}$$

$$\tau_t = \mu + \texttt{FFF} + \tau_{t-1} + \eta_t \tag{9}$$

$$\texttt{FFF} \equiv \sum_{k=1}^{n} a_k \sin(2\pi kt/T) + \sum_{k=1}^{n} b_k \cos(2\pi kt/T) \tag{10}$$

$$(1 - \phi_1 L - \phi_2 L^2)c_t = e_t \tag{11}$$

$$\begin{pmatrix} \eta_t \\ e_t \end{pmatrix} \sim \texttt{i.i.d.N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_\eta^2 & \sigma_{\eta e} \\ \sigma_{\eta e} & \sigma_e^2 \end{pmatrix}\right) \tag{12}$$

where y_t is 100 times the quarterly value of log real U.S. GDP, τ_t is the unobserved trend, and c_t is the cycle. Following the literature (for instance, see **?** and **?**) we let the cyclical component takes the AR(2) form ar2.

The trend and cycle innovations (η_t and e_t) are jointly normally distributed with mean zero and a general variance-covariance matrix. We follow **?** and do not restrict the covariance $\sigma_{\eta e}$ to be zero. One important issue is comparing the standard errors σ_η and σ_e, which measure the volatility of the shocks in trend and cycle, respectively. Because the trend τ_t is a random walk with drift, the trend shock η_t has permanent effect, and is commonly referred to as the real shock.

As mentioned in Introduction, the cycle will be ill-estimated if the trend is misspecified. In particular, this paper tries to account for possible structural breaks or smooth changes in the trend[2]. Toward that end, $\mu + \texttt{FFF}$ in model denotes the possibly time-varying drift term in the trend. In the presence of sharp or smooth structural change, some or all trigonometric terms in the FFF fff should be statistically significant. On the other hand, if $a_k = 0$, $b_k = 0$ for all k in fff, then our model is reduced to the unrestricted UC (URUC) model used by **?**[3]. Put differently, the URUC model assumes the drift term is a constant μ (and so there is no break) and our model nests the URUC model as a special case.

It is easy to see the link between our model and the PW model of **?**. The PW model

[2]It is possible that the trend is a nonlinear function of time without any breaks, and the nonlinearity is driven by some slow moving forces. We want to thank an anonymous reviewer for suggesting this possibility. In certain contexts, such as the Oil Crisis and Great Recession, it is conceptually straightforward to think of those events as possible sources of smooth or sharp structural changes, and link them to the estimated nonlinearity.

[3]The model is unrestricted in the sense that it does not restrict $\sigma_{\eta e}$ to be zero.

utilizes a dummy variable to represent the break of 1973 Oil Crisis, and the dummy variable equals zero before 1973Q1 and equals one thereafter. Mathematically, the PW model replaces fff with

$$d * \texttt{breakdummy}, \ \texttt{breakdummy} = \begin{cases} 0, & \text{before 1973Q1;} \\ 1, & \text{after 1973Q1.} \end{cases} \tag{13}$$

where d is the coefficient of the dummy variable for the break. The PW model implies a slowdown in GDP growth if the estimated d is negative.

Unlike the PW model, we do not need to assume that the form, number and/or location of the breaks are known. Instead, we let the "data talk" and use the FFF to approximate the breaks. As a practical matter, it is not desirable to use a large value of n. The use of many frequencies can exhaust degrees of freedom and can lead to over-fitting. Here we follow the principle of parsimony and the recommendation of Enders and Lee (2012), and estimate models with $n = 1$, $n = 2$ and $n = 3$ in fff. For each given n, the test for the null hypothesis $H_0 : a_k = b_k = 0, \forall k$, follows a standard χ^2 distribution. Rejecting the null hypothesis is equivalent to rejecting the use of a constant drift term in the trend specification. Given that H_0 can be rejected (so that there is a nonlinear trend), the recommendation is to select the value of n using standard model selection criteria.

Notice that the FFF fff is linear in the parameters a_k and b_k. As a result, our model is much easier to estimate than including the LSTAR or ESTAR function in the trend function model. Thus we can say our model is a computationally tractable way of allowing for gradual effects of structural change. Finally, our model is cast in state space form and estimated by the Kalman smoother. The Kalman smoother uses all information available in the sample, while the basic Kalman filter only uses information available up to time t. Thus, the Kalman smoother provides superior fit compared to the basic Kalman filter.

4. Estimation Results

4.1 Using MNZ Short Sample

We obtained the real U.S. GDP series from Federal Reserve Economic Data (FRED) [4]. In order to compare our FFF-UC model to that of ?, we begin by restricting the sample period to the period 1947Q1 - 1998Q2. Columns 2 and 3 of Table 1 report the results of Maximum Likelihood estimations[5] of the URUC model[6]. The estimates of the URUC model are close to those reported in ?. Note that the estimated standard deviation of the trend innovation σ_η is significant at the 5% level. The trend and cycle innovations are negatively correlated and their correlation coefficient is close to BN decomposition. In contrast, σ_η is insignificant and $\sigma_{\eta e}$ is positive in the PW model (shown in columns 4 and 5 of Table 1). The difference is due to the fact that PW model accounts for the break in 1973Q1 and the coefficient of the break dummy d is negative and significant.

Columns 6 through 11 of Table 1 report the results of estimating the model model with $n = 1, 2, 3$ in the FFF fff, respectively. Note that the Fourier model and PW model both indicate insignificant σ_η, and both imply positive correlation ($\sigma_{\eta e}$) of the trend and cycle innovations. Despite these similarities, the Fourier model differs from the PW model in the way we model the structural change. Note that a_1, the coefficient of the first trigonometric term, is significant in all three Fourier models. The Akaike Information Criterion (AIC) suggests that the Fourier model with three frequencies (Fourier 3) has the best fit, in which three out of six trigonometric terms (a_1, a_3 and b_3) are significant. This can be seen as the first evidence that at least one break is present in the trend[7].

[4] The website is `http://research.stlouisfed.org/fred2/data/GDPC1.txt`.

[5] The RATS codes are available upon request.

[6] The Hodrick—Prescott (HP) filter can also be used to approximate a smoothly evolving nonlinear trend. In addition to the well-known disadvantages of the HP filter, it does not readily allow for testing the null hypothesis of linearity (no breaks). Therefore in this paper we do not consider the HP filter.

[7] All models are best viewed as approximates for the unknown truth, and our model is no exception. We never know the break is smooth or not, nor do we know the number of breaks. Nevertheless, we hope to find statistical evidences indicating that one approximate (model) may outperform others. Here the AIC is used

One econometric issue is to determine whether it is possible to reject the null hypothesis of a linear trend in real U.S. GDP. We consider the likelihood ratio (LR) statistic:

$$\texttt{LR} = -2(\log L_{restricted} - \log L_{unrestricted}), \tag{14}$$

where $\log L_{restricted}$ is the log likelihood of the URUC model, which is restricted in this context because it imposes the null hypothesis of no structural change. The unrestricted models are the PW model and our Fourier models. Under the null hypothesis the LR test follows a Chi-squared distribution with degrees of freedom equal to the number of restrictions[8]. Table 1 shows that the hypothesis of no breaks is rejected at the 5% level in all cases.

We also report the Bayesian Information Criterion (BIC). A well known fact is that BIC tends to pick a model that is more parsimonious than AIC. Among the three Fourier models, BIC picks the Fourier 1 model. Thus, it is debatable which Fourier model is the best one, depending on which criterion to use. However, the key point is that the LR test indicates undoubtedly that there is a nonlinear trend that can be represented by the FFF. In this regard, our results complement **?** by considering the possibility of a nonlinear trend function.

Next we compare the trend-cycle decomposition obtained from the Fourier 1 model (selected by BIC) to those obtained from the URUC and PW models. Panel 1 of Figure 2 compares the Fourier 1 trend to URUC trend. Notice that the Fourier 1 trend suggests that accounting for the nonlinearity of the trend is important. Relative to the URUC trend, the shape of the Fourier 1 trend indicates that the productivity growth was strong (the trend was convex) during middle 60's, and weak (the trend was concave) in late 70s and early 80s. However, the resumption of productivity growth in 1990s is not obvious in the Fourier 1

for model comparisons.

[8]The Chi-squared distribution is valid because we compute the LR test for given frequency and for given break dummy. In other words, here we do not use the sup Wald test suggested by **?**, which follows a nonstandard distribution.

trend.

Panel 2 compares the Fourier 1 trend to PW trend. We find the two trends largely overlap. However, since the PW trend uses only one sharp break at 1973Q1, it is piecewise linear, indicating that productivity growth was constant during 1955-1972 and 1973-1998, respectively. Therefore, the PW trend does capture other periods during which the trend is generally deemed to be changing.

For the sake of completeness, Panels 3 and 4 draw the Fourier 3 trend (selected by AIC). The difference between Fourier 3 trend and URUC trend is similar to Panel 1. But the difference between the Fourier 3 trend and PW trend in Panel 4 is more obvious than Panel 2. For instance, the failure of the PW trend to capture the break in early 60s is clearly demonstrated in Panel 4. Moreover, Fourier 3 trend indicates that the slowdown of GDP growth may start in late 60s, and this finding is consistent with **?**.

Panels 5-8 draw the cycle components along with the recessions dated by NBER (shaded). It is shown that the Fourier cycle is generally smaller in magnitude than the PW cycle and URUC cycle[9]. The implication about the severity of recession is also different. Take the recession in 1982. The Fourier 1 cycle indicates the 1982 recession is less severe than that implied by the PW cycle and URUC cycle. The gap between the Fourier 3 cycle and PW cycle is most apparent in 60s. This result is expected since in that period there may be a break but the PW model rules out the possibility of its existence.

4.2 Using the Recent Data

Updating the results is particularly interesting because the MNZ short sample does not include the recessions of 2001 and 2009. As such, we reestimate all models using the extended sample that covers the 1947Q1-2013Q3 period. Table 2 reports the new results. Once again,

[9]We conduct a Monte Carlo simulation to investigate the possibility of "spurious cycle". We find our model is unlikely to produce spurious cycle relative to URUC model under the hypothesis that there is no break in trend. More explicitly, we compute the ratio of standard deviations of cyclical components of Fourier model to URUC model assuming no breaks in trend. Most often that ratio is close to unity.

the hypothesis of no breaks in the trend is rejected in all cases. Moreover, the PW model and Fourier models indicate insignificant σ_η and positive $\sigma_{\eta e}$.

Nevertheless, including the 2001 and 2009 recessions in the estimation leads to several noticeable changes. First, in the URUC model σ_η decreases from 1.06 to 0.44, and $\sigma_{\eta e}$ now becomes positive and insignificant. Second, in the PW model the coefficient of the break dummy falls from -0.20 to -0.26. That means the 2001 and 2009 recessions cause further slowdown in GDP growth, but the PW model mistakenly attributes it to the 1973 Oil crisis. Third, for the Fourier models, AIC picks the Fourier 2 as the best model. BIC still chooses the Fourier 1 model.

The effects of the recent two recessions can be best seen in Figure 3. In Panel 1, for instance, those two recessions effectively pull the URUC trend down. As a result, the Fourier 1 trend is almost always above URUC trend. That translates to the Fourier 1 cycle being consistently below URUC cycle in Panel 5. In our view, the Fourier 1 cycle seems more reasonable than the URUC cycle because the former appears "mean-reverting" while the latter shows persistent deviation from the value of zero.

Furthermore, the gap between the Fourier 1 cycle and PW cycle in Panel 6 is wider than the Panel 6 in Figure 2. So using additional recessions is really helpful for identification purpose. With more recessions, the results of the PW model and Fourier models become more distinguishable.

The Great Recession is especially important in that it can be used as a benchmark to evaluate the relevance of all models. According to the NBER chronology the Great Recession ended in 2009Q1. That turning point is captured only by the Fourier 2 cycle shown in Panels 7 and 8. In contrast, the URUC cycle and PW cycle both fail to predict the end of the Great Recession. Actually the URUC cycle and PW cycle imply that the recession would have worsened even after 2010. Therefore, in terms of matching with the latest data, the Fourier 2 model outperforms all other models.

To check the robustness and ensure the fairness in comparison, we estimate a new PW model with two dummy variables: the first one still represents the 1973Q1 break, and the second one uses 2008Q3 as the break date[10]. The estimated coefficients of the two dummy variables are -0.19 and -0.68 respectively, and both are significant at the 5% level[11]. The coefficient of the second dummy variable confirms that the Great Recession causes further slowdown in the GDP growth. Figure 4 compares the Fourier 2 cycle to the cycle obtained in the new PW model (denoted by PW2). Note that the PW2 cycle is able to capture the turning point where Great Recession ended. This in turn implies that capturing that turning point requires taking the Great Recession into account. The Fourier 2 model of course does that, but the URUC and the original PW models fail to do so.

A noticeable difference between the Fourier 2 cycle and PW2 cycle still exists, for example, in the period 1965-1969. This fact, along with the finding of **?**, casts doubt on using 1973Q1 as the break date. If the GDP slowdown really started in late 60s other than in 1973, then the PW2 cycle would overestimate the true cycle.

Another striking difference can be seen in the end of sample: the PW2 cycle indicates a strong and rapid recovery after the Great Recession. According to the PW2 cycle, the U.S. GDP would return to its potential level in 2012. The Fourier 2 cycle, however, predicts a much weaker and slower recovery, which is in line with the projection of Congressional Budget Office that U.S. GDP would not reach its potential level until 2017[12]. In light of this, the Fourier model outperforms the new PW model even though the new PW model has taken into account the Great Recession.

In this case the superiority of Fourier model may be attributed to the momentum or lingering effect of the Great Recession. Just because a recession ends does not necessarily

[10] The real GDP started contracting in 2008Q3. See `http://research.stlouisfed.org/fred2/data/GDPC1.txt`.

[11] There is no qualitative change in σ_η, σ_e and $\sigma_{\eta e}$ in the new PW model relative to the old PW model. Full estimation results of the new PW model are available.

[12] The report can be found at `http://www.cbo.gov/publication/43907`.

mean its effect will disappear instantaneously. The Fourier model involves trigonometric functions, and therefore is suitable for capturing the smooth effect of structural change.

5. Discussion

It is instructive to compare two tests for the null hypothesis of no structural breaks. We consider the Supremum Likelihood Ratio (Sup LR) Test of **?**, which is appropriate when the break location is unknown, and the Fourier LR test reported in Tables 1 and 2. The data generating processes are

$$y_t = 0.4I(t < \lambda T) + (0.4 + \Delta)I(t \geq \lambda T) + e_t, \quad \texttt{(Sharp Break)} \tag{15}$$

$$y_t = [1 + \exp(-\gamma(t - \lambda T))]^{-1}\Delta + e_t, \quad \texttt{(Smooth Break)} \tag{16}$$

where $I(\,)$ is the indicator function, e_t follows the standard normal iid process, and the sample size is $T = 200$. We assume there is at most one break in the expected value of y_t, and the break can be sharp or smooth (taking the logistic smooth transition function form). The key parameters are Δ, which measures the shift in the mean, λ, which measures the break fraction (λT is the break date), and γ, which measures how smooth the transition is.

For the Sup LR test we run the regression $y_t = \beta_0 + \beta_1 I(t \geq \lambda T) + u_t$ repeatedly for each $\lambda \in [\pi_0, 1 - \pi_0], \pi_0 = 0.10$. The Sup LR test is the maximum value of the LR test for the null hypothesis $H_0 : \beta_1 = 0$. This test follows nonstandard distribution under the null hypothesis, and the critical values are from Table 1 of **?**. For the Fourier LR test we let the number of frequencies be one. The regression is $y_t = \beta_0 + \beta_1 \sin(2\pi t/T) + \beta_2 \cos(2\pi t/T) + u_t$, and the null hypothesis is $H_0 : \beta_1 = \beta_2 = 0$. Conditional on this frequency number, the Fourier LR test follows $\chi^2(2)$ distribution if there is no break.

Table 3 reports the rejection rates of the two tests at the 5% level using 10,000 simulations. We obtain the size of the test when $\Delta = 0$ (there is no break); while we observe the power when $\Delta \neq 0$. Panels A and B use the sharp break shb, and Panels C, D and E use the smooth break smb.

There are three main findings. First, it seems that the Sup LR test is slightly under-sized, and the Fourier LR test has less size distortion. Second, as expected, the powers of both tests rise when Δ deviates from zero. Third, no test dominates the other in power. There is evidence that the Sup LR test has greater power than the Fourier LR test when λ is 0.8 (the break occurs in the end of the sample), and the power of the Sup LR test improves when γ rises (the transition becomes less smooth). Overall, Table 3 shows that the sizes and powers of the two tests are comparable.

Even though both tests can be used with unknown break dates, one benefit of the Fourier LR test is to allow for general functional forms of structural breaks. Moreover, the Fourier LR test is more convenient than the Sup LR test when the number of breaks is known. In the presence of several breaks, the Sup LR test entails a multi-dimensional grid search for the breaks dates, much harder than estimating multiple trigonometric terms required by the Fourier LR test.

Next we check whether the result of the URUC model reported in Table 2 is sensitive to initial values. Two log likelihood values of the URUC model are computed based on the 1947Q1-2013Q3 sample: one uses the parameter values of the URUC model reported in Table 4 of **?**, and the other uses the parameter values in Table 1 of this paper. The log likelihood values are -334.85 and -335.42, both being less than -333.94, the value reported in Table 2 for the URUC model[13].

6. Conclusion

This paper provides a new trend-cycle decomposition of U.S. real GDP. The focus is on accounting for possibly smooth structural change and/or multiple sharp breaks in the trend

[13]Recent study of **?** uses the 1947Q1-2011Q4 sample, and reports an estimated URUC model close to **?**.

component. It would be hard, if not impossible, to achieve this type of decomposition using the traditional methods. For example, **?** account for only one "exogenous" break using the dummy-variable approach. The estimation would become exponentially more difficult if one attempted to allow for multiple "endogenous" breaks using a grid search for the unknown break dates. The same is true if we try to allow for gradual (smooth) breaks using the conventional nonlinear models such as LSTAR and ESTAR. For one thing, just like the dummy-variable approach, those models necessitate knowing or estimating the break dates. It would soon become computationally intractable if we directly augment the trend component with the LSTAR or ESTAR function.

Our idea of using the Flexible Fourier Form (FFF) provides a simple solution to the problem of approximating a nonlinear trend. We have shown, both through simulation and real-data estimation, that FFF can successfully approximate various types of breaks, regardless of their number, form, and location. It is especially important that the method allows the trend breaks to be smooth or sharp. The FFF has three distinguishing features. First, it provides a global rather than local approximation for the types of nonlinearities likely to be present in real GDP data. Second, the FFF is linear in the unknown parameters for given frequency, and that makes FFF much easier to estimate than the LSTAR and ESTAR models. Thirdly, hypothesis testing is facilitated because there is not a nuisance parameter problem and the Fourier frequencies are orthogonal to each other. We have shown a small number of low frequencies are sufficient to capture very complicated patterns in the trend.

After applying the FFF decomposition to the recent sample of 1947Q1-2013Q3, we find that (i) the hypothesis of a linear trend in in U.S. GDP can be rejected; (ii) the standard deviation of the trend innovation is insignificant; (iii) the innovations of trend and cycle are positively correlated (hence, instead of shocks causing the trend and cycle to move in opposite directions, our model finds that negative shocks result in simultaneous declines in both the trend and the cycle); (iv) the FFF cycle matches the NBER dating very well, especially for the Great Recession; (v) the new PW model that accounts for both 1973 and 2008 breaks implies a rapid recovery after the Great Recession, whereas our Fourier model indicates a slow recovery.

We expect that the FFF methodology can be applied in other settings whenever breaks or other types of nonlinearities need to be accounted for. One potential topic is to modify **?** and use the FFF to approximate smooth breaks in the conditional volatility of financial time series. The dummy-variable approach of **?** imposes an upper bound for the number of breaks, which may be too restrictive for a high-frequency financial time series with many observations. In that case the FFF approach can be less restrictive and computationally more feasible.

Table 1: Estimation Results, Short Sample (1947Q1-1998Q2)

	URUC		PW		Fourier 1		Fourier 2		Fourier 3	
	Est	s.e	Est	s.e	Est	s.e	Est	s.e	Est	s.e
μ	0.87*	(0.06)	0.96*	(0.02)	0.86*	(0.01)	0.86*	(0.02)	0.88*	(0.01)
σ_η	1.06*	(0.07)	0.00	(0.14)	0.00	(0.12)	0.00	(0.12)	0.00	(0.07)
σ_e	1.22*	(0.16)	0.43*	(0.13)	0.44*	(0.21)	0.41*	(0.14)	0.29	(0.18)
$\sigma_{\eta e}$	-0.95*	(0.14)	0.36*	(0.07)	0.36*	(0.12)	0.37*	(0.10)	0.41*	(0.10)
ϕ_1	1.34*	(0.09)	1.51*	(0.08)	1.51*	(0.09)	1.52*	(0.11)	1.53*	(0.10)
ϕ_2	-0.40*	(0.06)	-0.58*	(0.08)	-0.58*	(0.09)	-0.60*	(0.10)	-0.64*	(0.10)
a_1					0.13*	(0.03)	0.13*	(0.03)	0.13*	(0.02)
b_1					0.02	(0.04)	0.01	(0.04)	0.06	(0.03)
a_2							-0.06	(0.05)	-0.06	(0.03)
b_2							-0.00	(0.06)	0.04	(0.04)
a_3									0.15*	(0.05)
b_3									0.18*	(0.06)
d			-0.20*	(0.04)						
$\log L$	-267.3		-261.0		-261.4		-260.8		-254.4	
LR			12.5*		11.7*		12.9*		25.8*	
AIC	2.65		2.60		2.62		2.63		2.59	
BIC	2.75		2.72		2.75		2.79		2.78	

Note:
a: * denotes significance at the 5% level.
b: URUC model denotes the unrestricted unobserved component model used by **?**; PW model is the model used by **?**; Fourier 1, 2, 3 denote our new model model with $n = 1, 2, 3$ in the FFF fff respectively.
c: the sample is 1947Q1-1998Q2. The GDP data are from `http://research.stlouisfed.org/fred2/data/GDPC1.txt`.

Table 2: Estimation Results, Extended Sample (1947Q1-2013Q3)

	URUC		PW		Fourier 1		Fourier 2		Fourier 3	
	Est	s.e	Est	s.e	Est	s.e	Est	s.e	Est	s.e
μ	0.82*	(0.05)	0.97*	(0.04)	0.80*	(0.02)	0.78*	(0.02)	0.79*	(0.02)
σ_η	0.44*	(0.03)	0.00	(0.21)	0.00	(0.21)	0.00	(0.10)	0.00	(0.10)
σ_e	0.65*	(0.20)	0.48*	(0.21)	0.46*	(0.13)	0.33	(0.19)	0.36	(0.25)
$\sigma_{\eta e}$	0.07	(0.12)	0.28*	(0.13)	0.30*	(0.08)	0.35*	(0.08)	0.34*	(0.12)
ϕ_1	1.52*	(0.11)	1.51*	(0.10)	1.54*	(0.09)	1.57*	(0.09)	1.56*	(0.07)
ϕ_2	-0.53*	(0.11)	-0.55*	(0.09)	-0.57*	(0.09)	-0.63*	(0.09)	-0.62*	(0.07)
a_1					0.15*	(0.04)	0.14*	(0.02)	0.14*	(0.02)
b_1					-0.03	(0.05)	-0.07	(0.04)	-0.06	(0.04)
a_2							0.11*	(0.04)	0.11*	(0.03)
b_2							-0.15*	(0.05)	-0.15*	(0.05)
a_3									0.07	(0.06)
b_3									0.03	(0.06)
d			-0.26*	(0.06)						
$\log L$	-333.9		-329.0		-330.3		-325.4		-324.7	
LR Test			9.85*		7.28*		17.00*		18.58*	
AIC	2.55		2.52		2.53		2.51		2.52	
BIC	2.63		2.61		2.64		2.65		2.68	

Note:
a: * denotes significance at the 5% level.
b: URUC model denotes the unrestricted unobserved component model used by ?; PW model is the model used by ?; Fourier 1, 2, 3 denote our new model model with $n = 1, 2, 3$ in the FFF fff respectively.
c: the sample is 1947Q1-2013Q3. The GDP data are from `http://research.stlouisfed.org/fred2/data/GDPC1.txt`.

Table 3: Rejection Rates of Fourier LR Test and Sup LR Test

Panel A

$\lambda = 0.5$	$\Delta = -0.4$	$\Delta = -0.2$	$\Delta = 0$	$\Delta = 0.2$	$\Delta = 0.4$	$\Delta = 0.6$
Fourier LR Test	0.62	0.20	0.05	0.20	0.62	0.94
Sup LR Test	0.59	0.16	0.04	0.16	0.59	0.94

Panel B

$\lambda = 0.8$	$\Delta = -0.4$	$\Delta = -0.2$	$\Delta = 0$	$\Delta = 0.2$	$\Delta = 0.4$	$\Delta = 0.6$
Fourier LR Test	0.26	0.10	0.05	0.10	0.25	0.51
Sup LR Test	0.38	0.11	0.04	0.10	0.38	0.76

Panel C

$\gamma = 0.1, \lambda = 0.5$	$\Delta = -0.4$	$\Delta = -0.2$	$\Delta = 0$	$\Delta = 0.2$	$\Delta = 0.4$	$\Delta = 0.6$
Fourier LR Test	0.55	0.18	0.05	0.18	0.55	0.90
Sup LR Test	0.53	0.15	0.04	0.14	0.53	0.89

Panel D

$\gamma = 0.2, \lambda = 0.5$	$\Delta = -0.4$	$\Delta = -0.2$	$\Delta = 0$	$\Delta = 0.2$	$\Delta = 0.4$	$\Delta = 0.6$
Fourier LR Test	0.60	0.18	0.06	0.19	0.60	0.92
Sup LR Test	0.56	0.15	0.04	0.16	0.57	0.92

Panel E

$\gamma = 0.2, \lambda = 0.8$	$\Delta = -0.4$	$\Delta = -0.2$	$\Delta = 0$	$\Delta = 0.2$	$\Delta = 0.4$	$\Delta = 0.6$
Fourier LR Test	0.25	0.10	0.05	0.09	0.25	0.49
Sup LR Test	0.35	0.10	0.03	0.10	0.35	0.71

Note:
The rejection rates of the Sup LR and Fourier LR tests at the 5% level are reported based on 10,000 simulations.