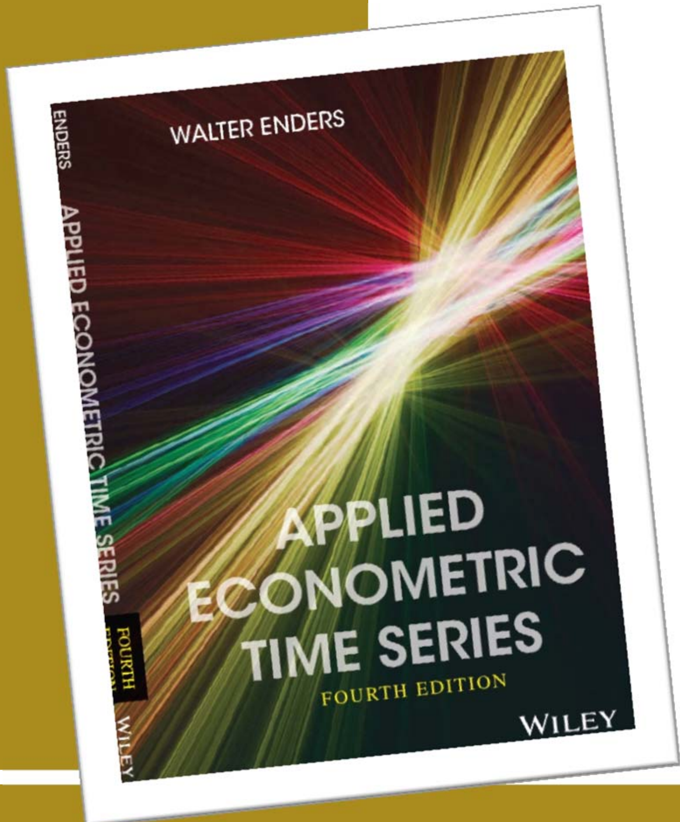

APPLIED ECONOMETRIC TIME SERIES FOURTH EDITION

Chapter 1: Difference Equations

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Section 1

TIME-SERIES MODELS

The traditional use of time series models was for forecasting

If we know

$$y_{t+1} = a_0 + a_1 y_t + \varepsilon_{t+1}$$

then

$$E_t y_{t+1} = a_0 + a_1 y_t$$

and since

$$\begin{aligned} y_{t+2} &= a_0 + a_1 y_{t+1} + \varepsilon_{t+2} \\ E_t y_{t+2} &= a_0 + a_1 E_t y_{t+1} \\ &= a_0 + a_1(a_0 + a_1 y_t) \\ &= a_0 + a_1 a_0 + (a_1)^2 y_t \end{aligned}$$

Capturing Dynamic Relationships

- With the advent of modern dynamic economic models, the newer uses of time series models involve
  - Capturing dynamic economic relationships
  - Hypothesis testing
- Developing "stylized facts"
  - In a sense, this reverses the so-called scientific method in that modeling goes from developing models that follow from the data.

The Random Walk Hypothesis

$$y_{t+1} = y_t + \varepsilon_{t+1}$$

or

$$\Delta y_{t+1} = \varepsilon_{t+1}$$

where y_t = the price of a share of stock on day t, and ε_{t+1} = a random disturbance term that has an expected value of zero.

Now consider the more general stochastic difference equation

$$\Delta y_{t+1} = a_0 + a_1 y_t + \varepsilon_{t+1}$$

The random walk hypothesis requires the testable restriction:

$$a_0 = a_1 = 0.$$

The Unbiased Forward Rate (UFR) hypothesis

Given the UFR hypothesis, the forward/spot exchange rate relationship is:

$$s_{t+1} = f_t + \varepsilon_{t+1} \qquad (1.6)$$

where ε_{t+1} has a mean value of zero from the perspective of time period t.

Consider the regression

$$s_{t+1} = a_0 + a_1 f_t + \varepsilon_{t+1}$$

The hypothesis requires $a_0 = 0$, $a_1 = 1$, and that the regression residuals ε_{t+1} have a mean value of zero from the perspective of time period t.

The spot and forward markets are said to be in *long-run equilibrium* when $\varepsilon_{t+1} = 0$. Whenever s_{t+1} turns out to differ from f_t, some sort of adjustment must occur to restore the equilibrium in the subsequent period. Consider the adjustment process

$$s_{t+2} = s_{t+1} - a[\, s_{t+1} - f_t \,] + \varepsilon_{st+2} \quad a > 0 \qquad (1.7)$$

$$f_{t+1} = f_t + b\,[\, s_{t+1} - f_t \,] + \varepsilon_{ft+1} \quad b > 0 \qquad (1.8)$$

where ε_{st+2} andεe_{ft+1} both have an expected value of zero.

Trend-Cycle Relationships

- We can think of a time series as being composed of:

$$y_t = \text{trend} + \text{“cycle”} + \text{noise}$$

  - Trend: Permanent
  - Cycle: predictable (albeit temporary)
    - (Deviations from trend)
  - Noise: unpredictable

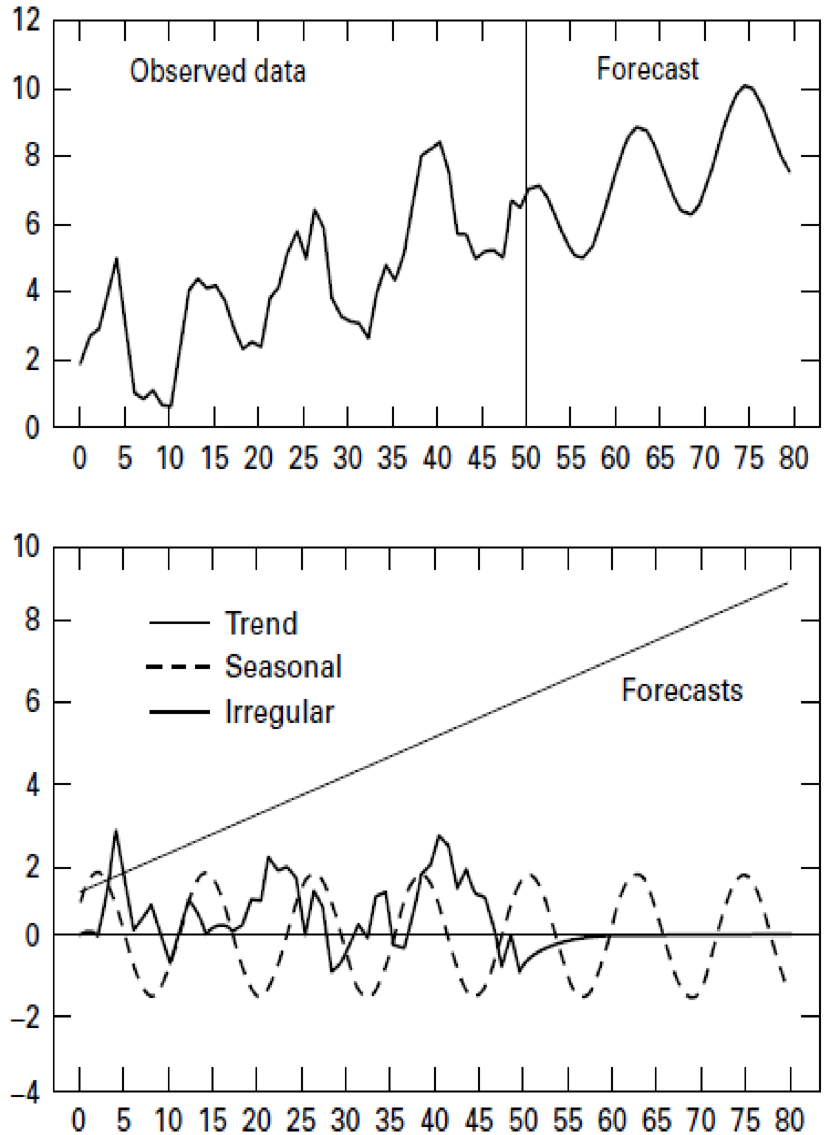

FIGURE 1.1 Hypothetical Time Series

Series with decidedly upward trend

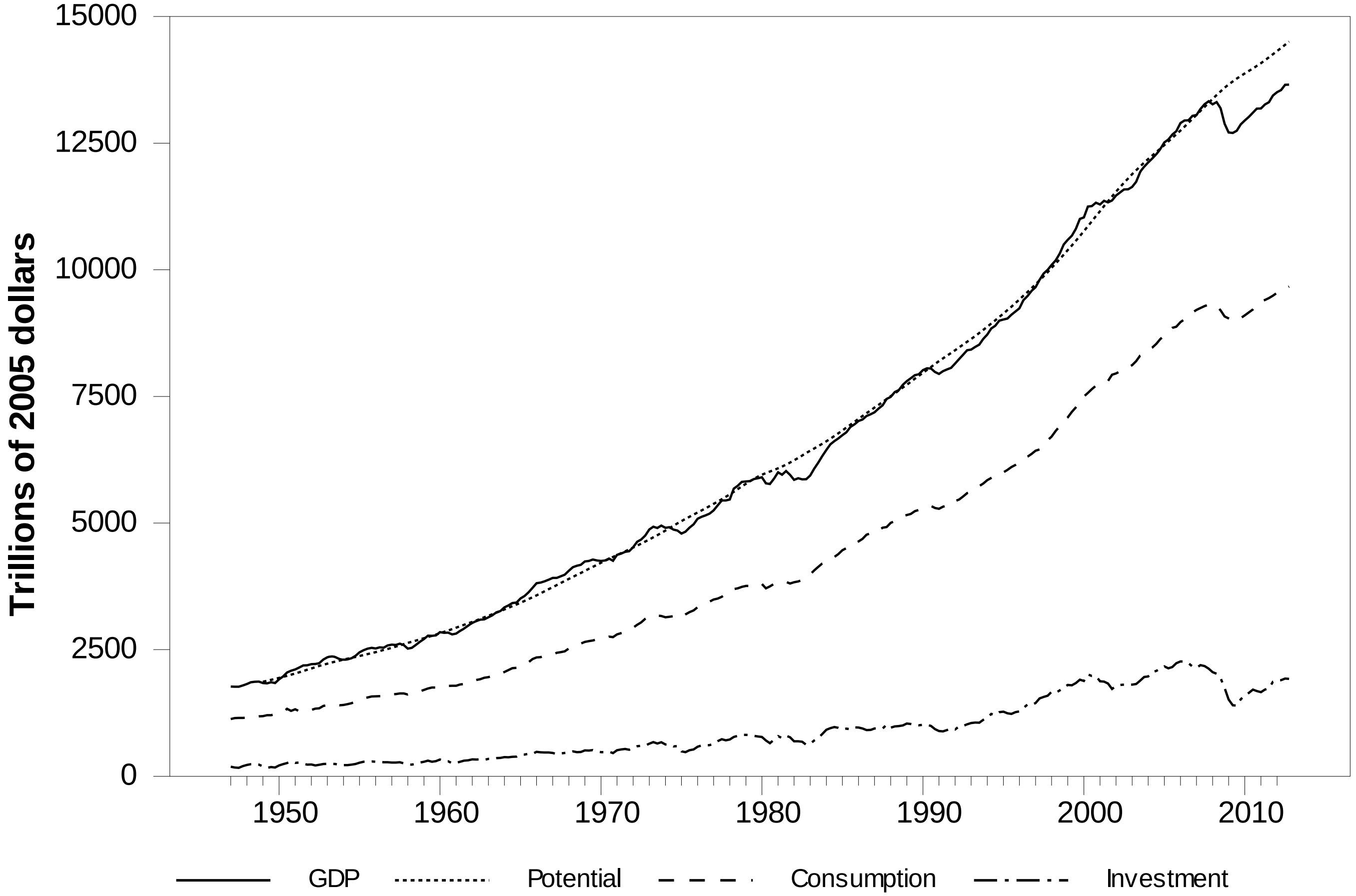

Figure 3.1 Real GDP, Consumption and Investment

GDP Volatility?

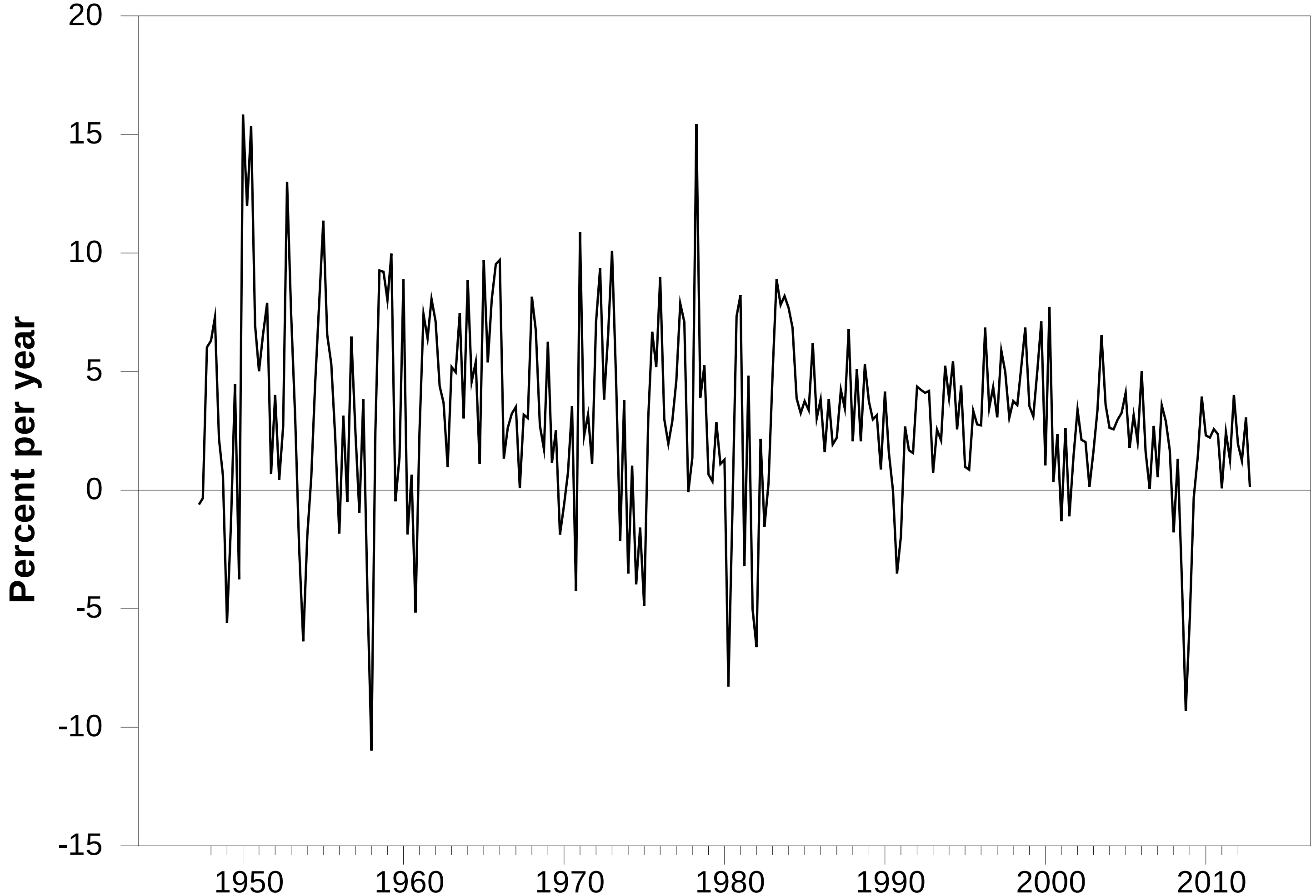

Figure 3.2 Annualized Growth Rate of Real GDP

Stock Market Volatility

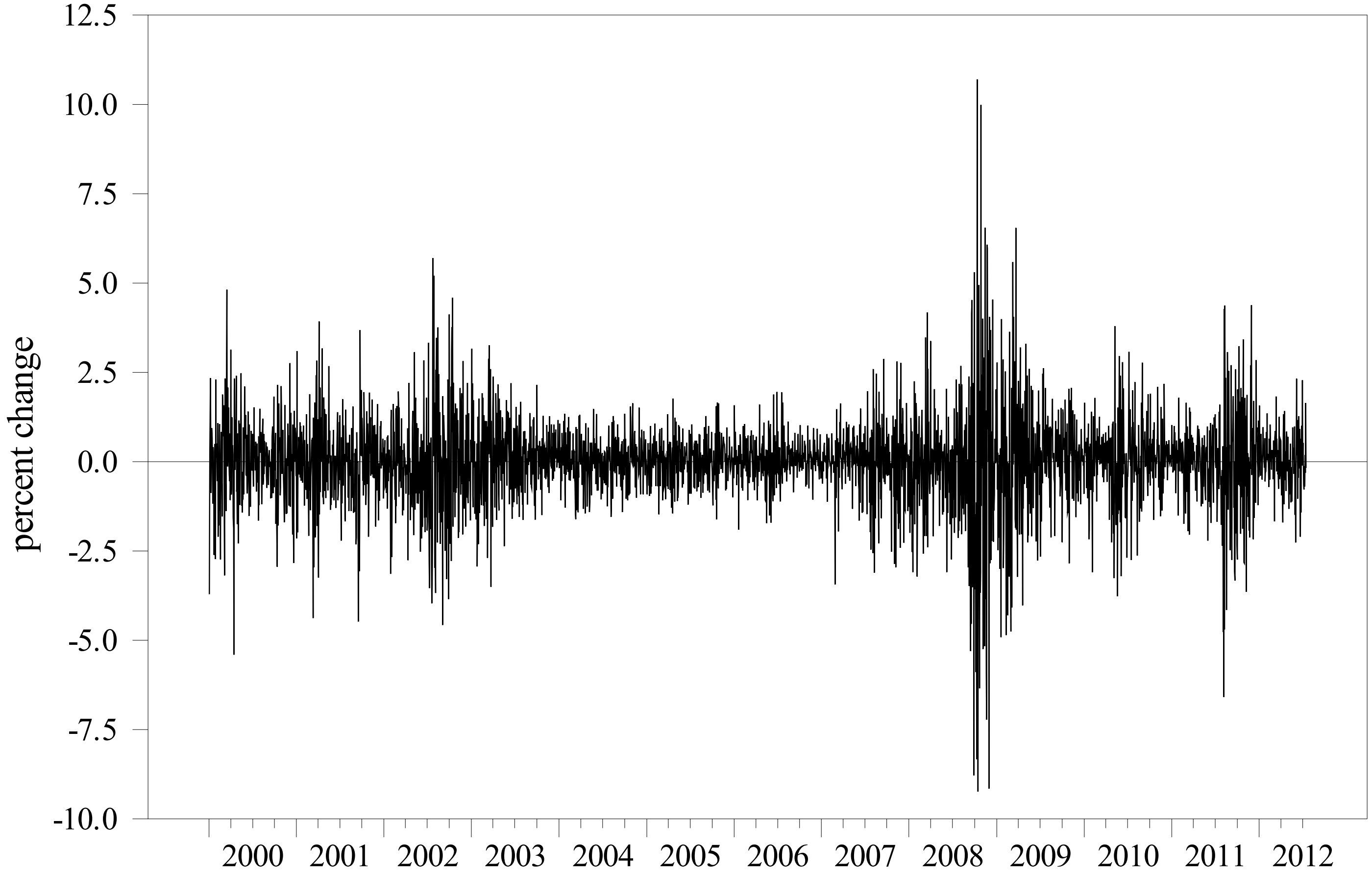

Figure 3.3: Daily Changes in the NYSE US 100 Index: (Jan 4, 2000 - July 16, 2012)

Co-Movements

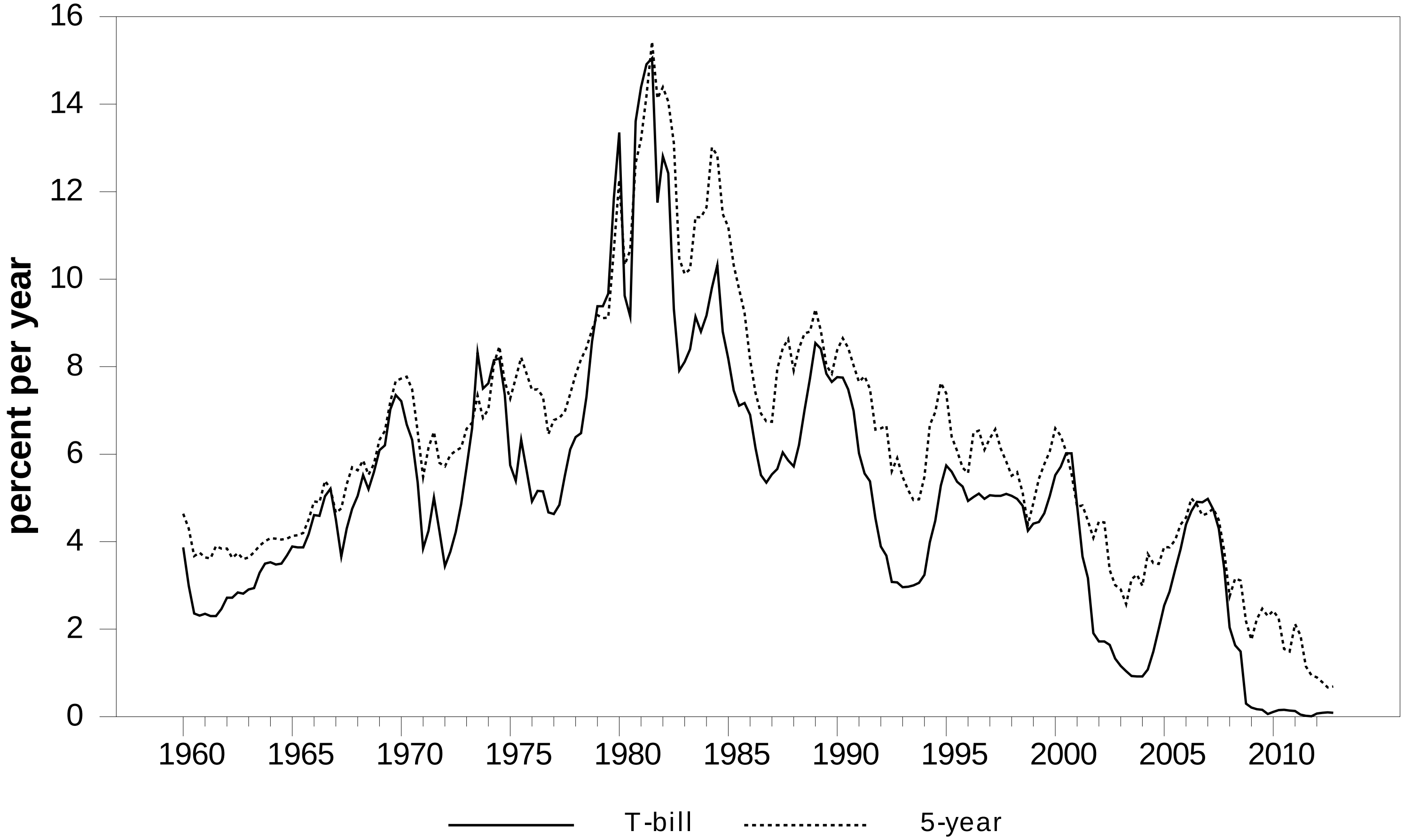

Figure 3.4 Short- and Long-Term Interest Rates

Common Trends

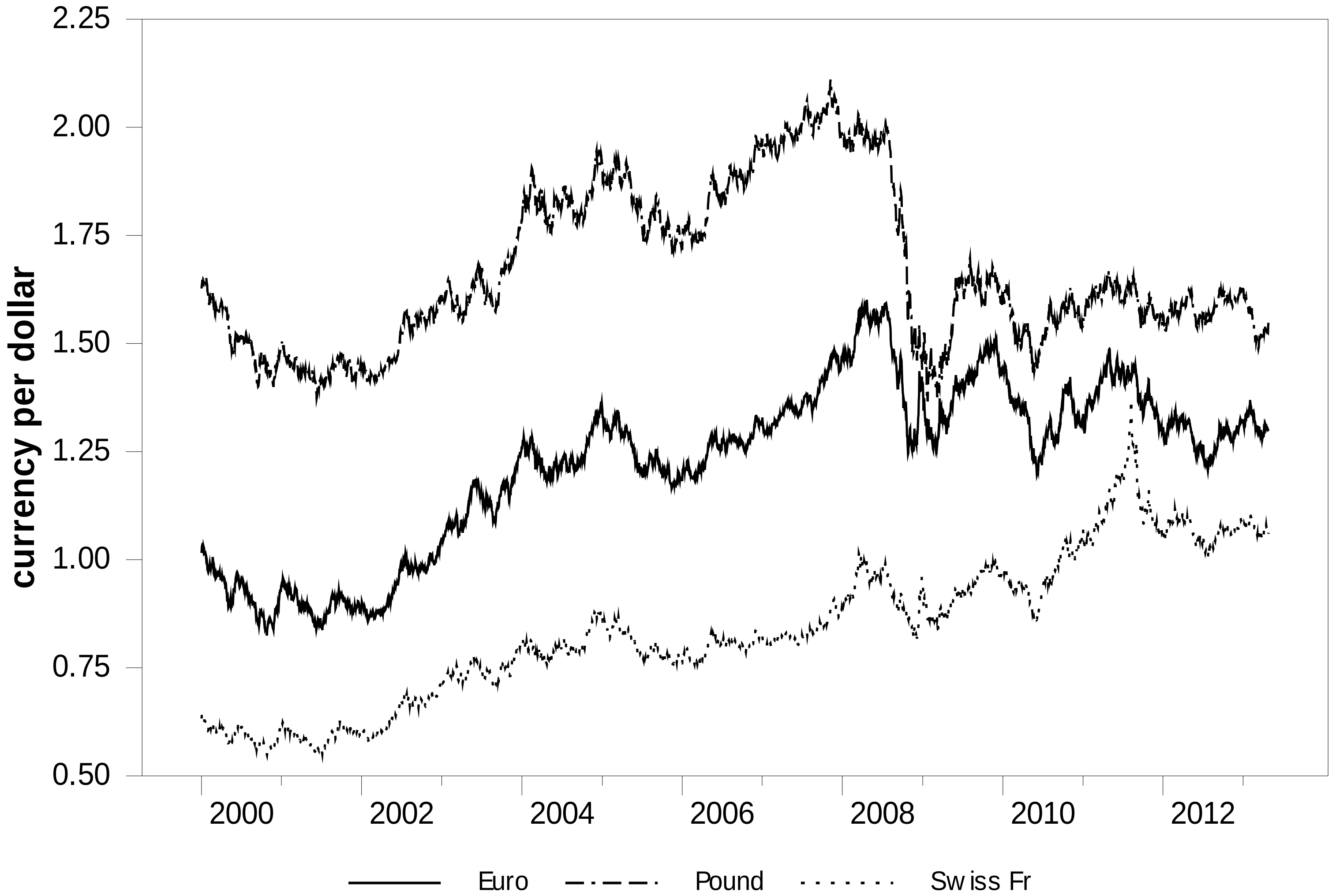

Figure 3.5: Daily Exchange Rates (Jan 3, 2000 - April 4, 2013)

Section 2

DIFFERENCE EQUATIONS AND THEIR SOLUTIONS

Consider the function $y_{t*} = f(t^*)$

$$\Delta y_{t^*+h} \equiv f(t^*+h) - f(t^*)$$
$$\equiv y_{t^*+h} - y_{t^*}$$

We can then form the **first differences:**

$$\Delta y_t = f(t) - f(t-1) \equiv y_t - y_{t-1}$$
$$\Delta y_{t+1} = f(t+1) - f(t) \equiv y_{t+1} - y_t$$
$$\Delta y_{t+2} = f(t+2) - f(t+1) \equiv y_{t+2} - y_{t+1}$$

More generally, for the forcing process x_t a n-th order linear process is

$$y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t$$

What is a solution?

A **solution** to a difference equation expresses the value of y_t as a function of the elements of the $\{x_t\}$ sequence and t (and possibly some given values of the $\{y_t\}$ sequence called **initial conditions**).

$$y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t$$

The key property of a solution is that it satisfies the difference equation for all permissible values of t and $\{x_t\}$.

Section 3

- Iteration without an Initial Condition
- Reconciling the Two Iterative Methods
- Nonconvergent Sequences

SOLUTION BY ITERATION

Solution by Iteration

Consider the first-order equation

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t \qquad (1.17)$$

Given the value of y_0, it follows that y_1 will be given b

$$y_1 = a_0 + a_1 y_0 + \varepsilon_1$$

In the same way, y_2 must be

$$\begin{aligned} y_2 &= a_0 + a_1 y_1 + \varepsilon_2 \\ &= a_0 + a_1[a_0 + a_1 y_0 + \varepsilon_1] + \varepsilon_2 \\ &= a_0 + a_0 a_1 + (a_1)^2 y_0 + a_1 \varepsilon_1 + \varepsilon_2 \end{aligned}$$

Continuing the process in order to find y_3, we obtain

$$\begin{aligned} y_3 &= a_0 + a_1 y_2 + \varepsilon_3 \\ &= a_0[1 + a_1 + (a_1)^2] + (a_1)^3 y_0 + a_1{}^2 \varepsilon_1 + a_1 \varepsilon_2 + \varepsilon_3 \end{aligned}$$

From

$$y_3 = a_0[1 + a_1 + (a_1)^2] + (a_1)^3 y_0 + a_1{}^2\varepsilon_1 + a_1\varepsilon_2 + \varepsilon_3$$

you can verify that for all $t > 0$, repeated iteration yields

$$y_t = a_0 \sum_{i=0}^{t-1} a_i^i + a_1^t y_0 + \sum_{i=0}^{t-1} a_i^i \varepsilon_{t-i}$$

If $| a_1 | < 1$, in the limit

$$y_t = a_0 /(1 - a_1) + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$$

Backwards Iteration

Iteration from y_t back to y_0 yields exactly the formula given by (1.18).

Since $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$, it follows that

$$y_t = a_0 + a_1 [a_0 + a_1 y_{t-2} + \varepsilon_{t-1}] + \varepsilon_t$$

$$= a_0(1 + a_1) + a_1\varepsilon_{t-1} + \varepsilon_t + a_1^2[a_0 + a_1 y_{t-3} + \varepsilon_{t-2}]$$

If $|a_1| < 1$, in the limit

$$y_t = a_0/(1 - a_1) + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$$

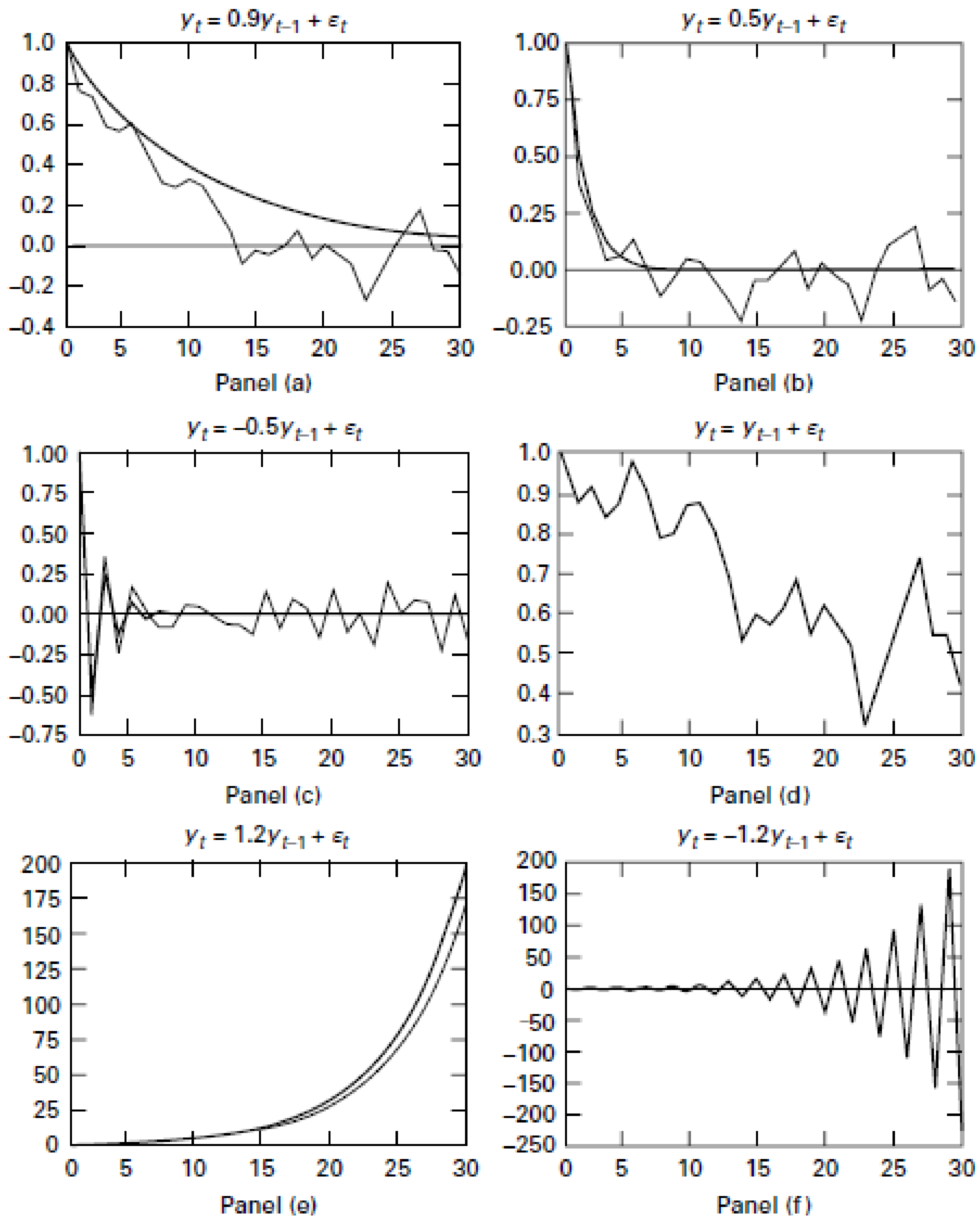

FIGURE 1.2 Convergent and Nonconvergent Sequences

Section 4

- The Solution Methodology
- Generalizing the Method

AN ALTERNATIVE SOLUTION METHODOLOGY

Does it converge? Characteristic Roots

Since the intercept and the ε_t sequence have nothing to do with the issue of convergence, consider:

$$y_t = a_1 y_{t-1}$$

A solution is $\quad y_t = A(a_1)^t$

Proof: $\quad Aa_1^t = a_1 A a_1^{t-1}$

- If $|a_1| < 1$, the y_t converges to zero as t approaches infinity. Convergence is direct if $0 < a_1 < 1$ and oscillatory if $-1 < a_1 < 0$.
- If $|a_1| > 1$, the homogeneous solution is not convergent. If $a_1 > 1$, y_t approaches ∞ as t increases. If $a_1 < -1$, the y_t oscillates explosively.
- If $a_1 = 1$, any arbitrary constant A satisfies the homogeneous equation $y_t = y_{t-1}$. If $a_1 = -1$, the system is *meta-stable*: $= 1$ for even values of t and -1 for odd values of t.

Generalizing the Method

$$y_t = \sum_{i=1}^{n} a_i y_{t-i}$$

$$A\alpha^{t} - a_1 A\alpha^{t\text{-}1} - a_2 A\alpha^{t\text{-}2} - \ \ldots - \ a_n A\alpha^{t\text{-}n} = 0$$

There are *n* roots

In the *n*th-order case

$$y_t^h = A_1(\alpha_1)^t + A_2(\alpha_2)^t + \ldots$$

For convergence, all of the roots must be less than unity in absolute value (or inside the unit circle if complex).

In an nth-order equation, a necessary condition for all characteristic roots to lie inside the unit circle is

$$\sum_{i=1}^{n} a_i < 1$$

Since the values of the a_i can be positive or negative, a sufficient condition for all characteristic roots to lie inside the unit circle is

$$\sum_{i=1}^{n} |a_i| < 1$$

At least one characteristic root equals unity if

$$\sum_{i=1}^{n} a_i = 1$$

Any sequence that contains one or more characteristic roots that equal unity is called a **unit root** process.

For a third-order equation, the stability conditions can be written as

$1 - a_1 - a_2 - a_3 > 0$

$1 + a_1 - a_2 + a_3 > 0$

$1 - a_1 a_3 + a_2 - a_3^2 > 0$

$3 + a_1 + a_2 - 3a_3 > 0$ or $3 - a_1 + a_2 + 3a_3 > 0$

Given that the first three inequalities are satisfied, one of the last conditions is redundant.

The Solution Methodology

STEP 1: form the homogeneous equation and find all *n* homogeneous solutions;

STEP 2: find a particular solution;

STEP 3: obtain the general solution as the sum of the particular solution and a linear combination of all homogeneous solutions;

STEP 4: eliminate the arbitrary constant(s) by imposing the initial condition(s) on the general solution.

Section 5

- Stability Conditions
- Higher-Order Systems

THE COBWEB MODEL

Setting supply equal to demand:

$$b + \beta p_{t-1} + \varepsilon_t = a - \gamma p_t$$

or

$$p_t = (-\beta/\gamma)p_{t-1} + (a - b)/\gamma - \varepsilon_t/\gamma$$

The homogeneous equation is $p_t = (-\beta/\gamma)p_{t-1}$.

If the ratio β/γ is less than unity, you can iterate (1.39) backward from p_t to verify that the particular solution for the price is

$$p_t^p = \frac{a-b}{\gamma+\beta} - \frac{1}{\gamma}\sum_{i=0}^{\infty}(-\beta/\gamma)^i \varepsilon_{t-i}$$

Stability requires $| \beta/\gamma | < 1$

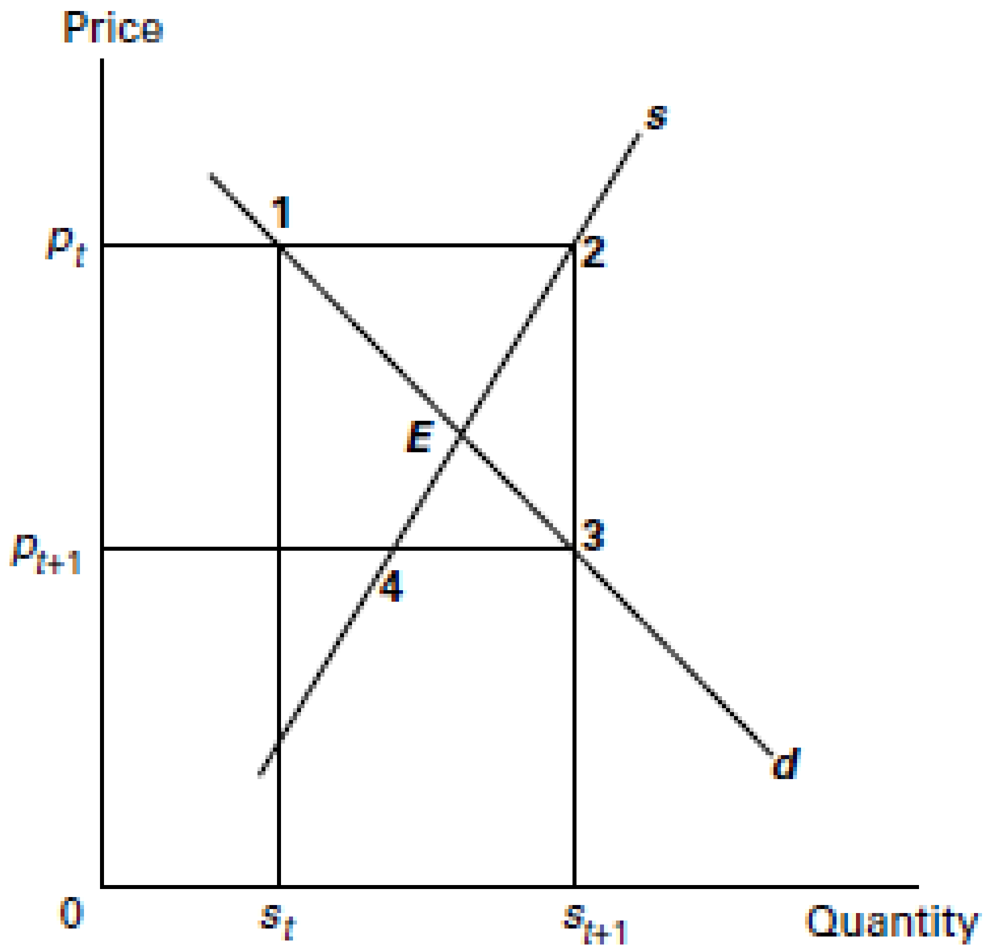

FIGURE 1.3 The Cobweb Model

Section 6

- Stability Conditions
- Higher-Order Systems

SOLVING HOMOGENEOUS DIFFERENCE EQUATIONS

Consider the second-order equation

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0$$

$$A\alpha^t - a_1 A\alpha^{t\text{-}1} - a_2 A\alpha^{t\text{-}2} = 0$$

If you divide (1.46) by $A\alpha^{t-2}$, the problem is to find the values of α that satisfy

$$\alpha^2 - a_1\alpha - a_2 = 0$$

There are two characteristic roots. Hence the homogeneous solution is

$$A_1(\alpha_1)^t + A_2(\alpha_2)^t$$

THE THREE CASES

CASE 1

If $a_1^2 + 4a_2 > 0$, d is a real number and there will be two distinct real characteristic roots.

CASE 2

If $+ 4a_2 = 0$, it follows that $d = 0$ and $a_1 = a_2 = a_1/2$. A homogeneous solution is $a_1/2$. However, when $d = 0$, there is a second homogeneous solution given by $t(a_1/2)^t$.

CASE 3

If $a_1^2 + 4a_2 < 0$, it follows that d is negative so that the characteristic roots are imaginary.

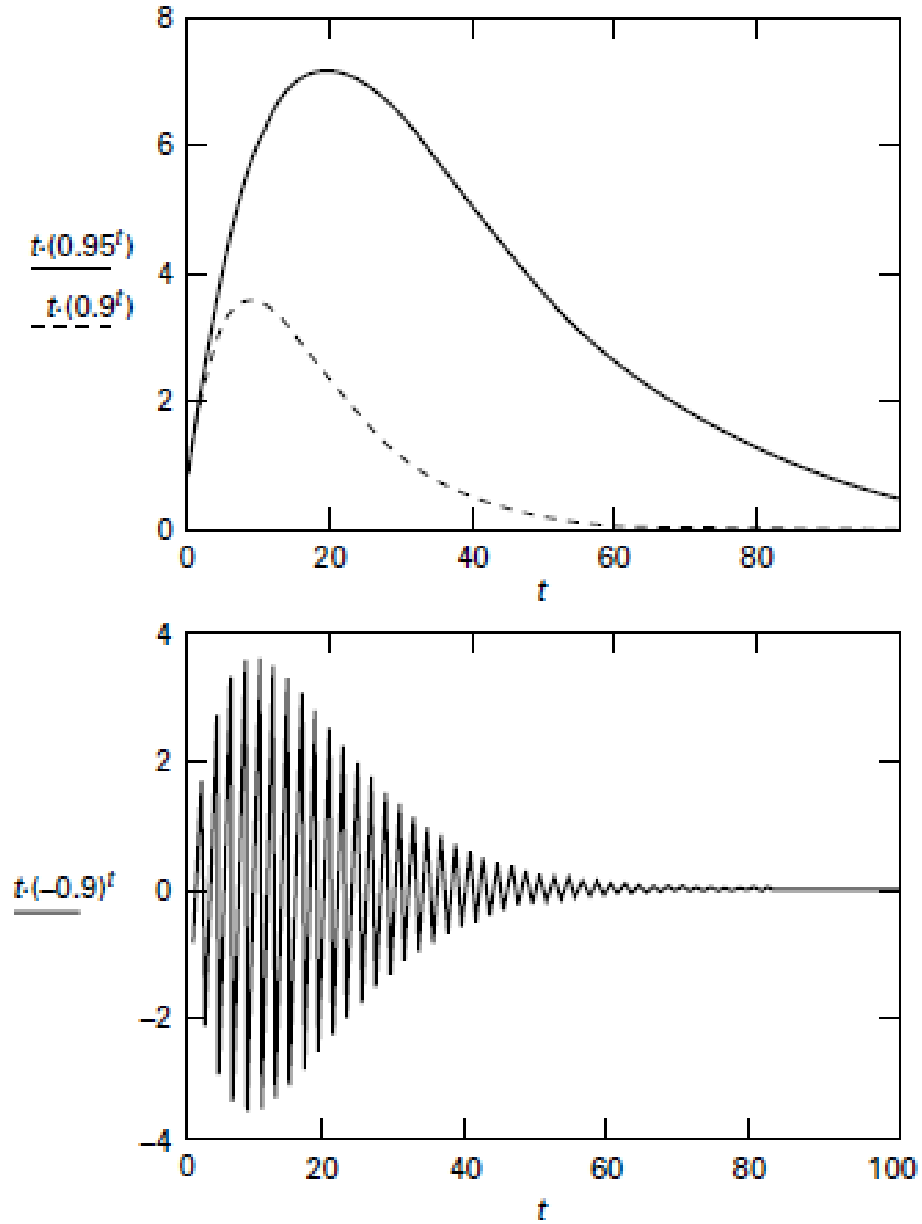

FIGURE 1.4 The Homogeneous Solution of $t(a_1)^t$

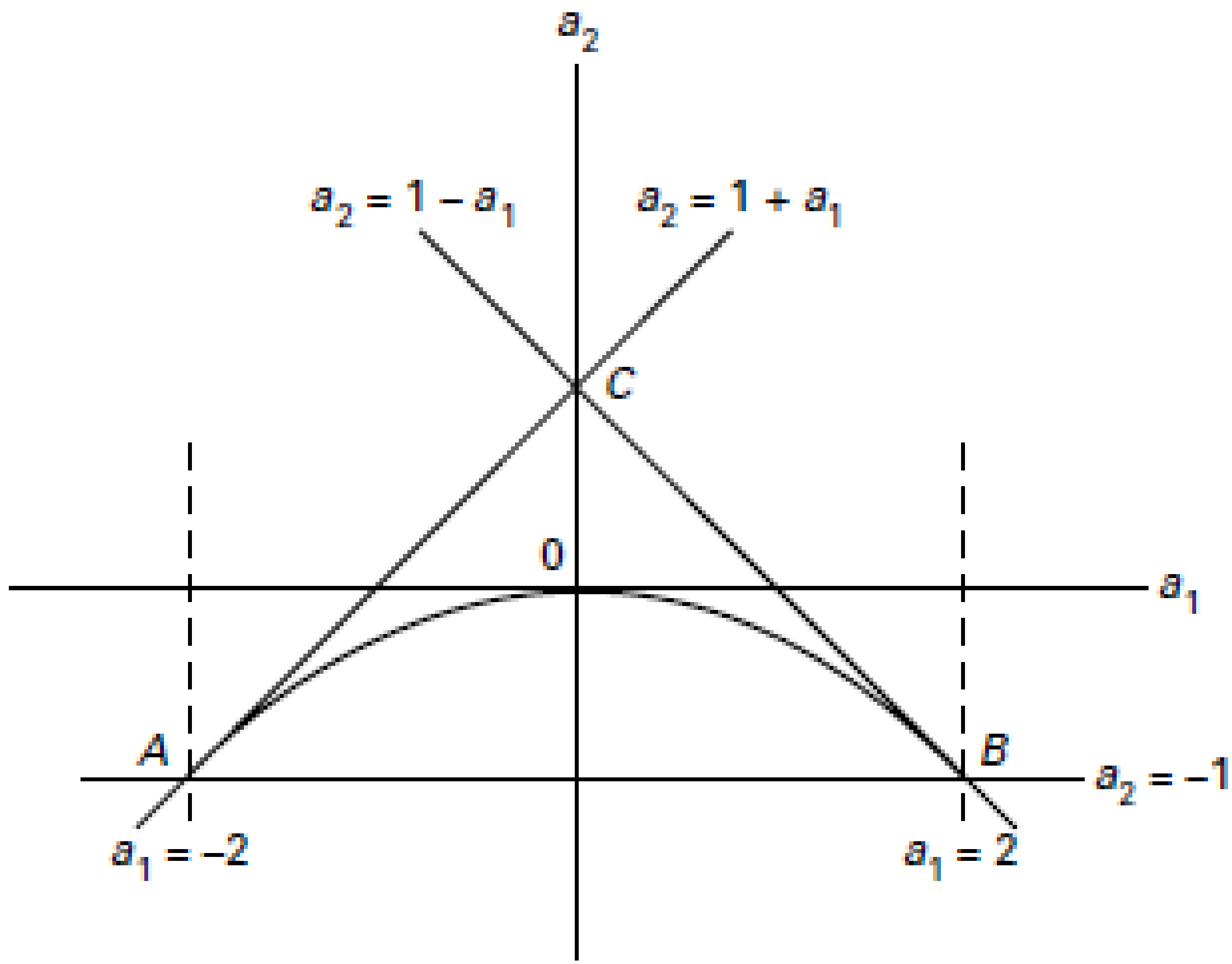

FIGURE 1.5 Characterizing the Stability Conditions

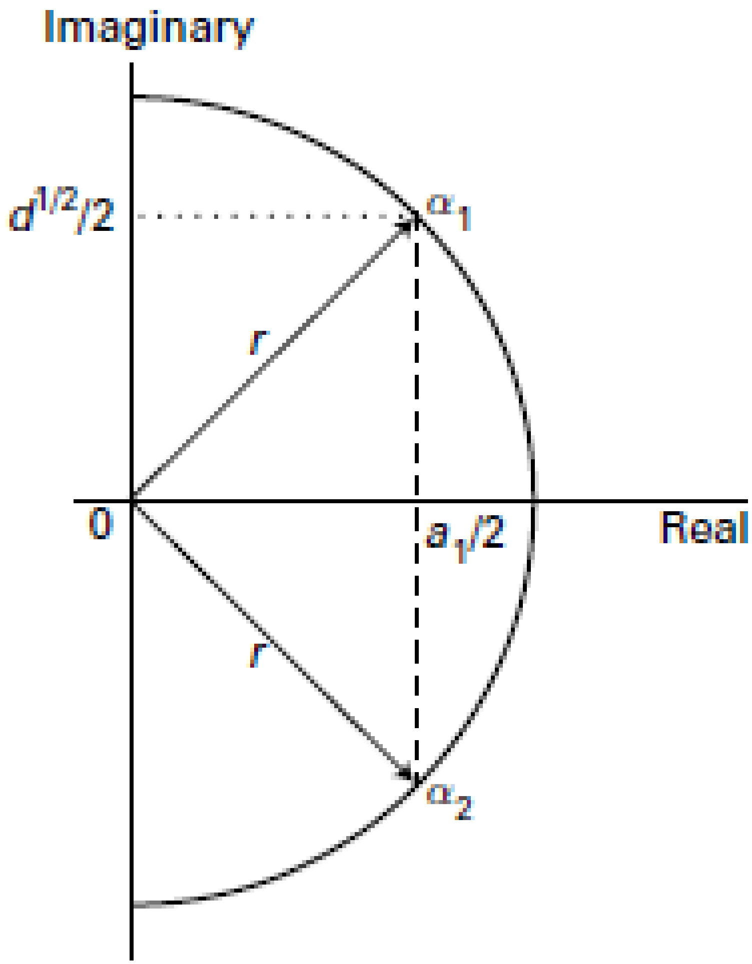

FIGURE 1.6 Characteristic Roots and the Unit Circle

WORKSHEET 1.1: SECOND-ORDER EQUATIONS

Example 1: $y_t = 0.2y_{t-1} + 0.35y_{t-2}$. Hence: $a_1 = 0.2$ and $a_2 = 0.35$

Form the homogeneous equation: $y_t - 0.2y_{t-1} - 0.35y_{t-2} = 0$

$d = \ + 4a_2$ so that $d = 1.44$. Given that $d > 0$, the roots will be real and distinct. Substitute $y_t = \alpha^t$ into the homogenous equation to obtain: $\alpha^t - 0.2\alpha^{t-1} - 0.35\alpha^{t-2} = 0$

Divide by α^{t-2} to obtain the characteristic equation: $\alpha^2 - 0.2\alpha - 0.35 = 0$

Compute the two characteristic roots: $\alpha_1 = 0.7$ $\alpha_2 = -0.5$

The homogeneous solution is: $A_1(0.7)^t + A_2(-0.5)^t$.

Example 2: $y_t = 0.7y_{t-1} + 0.35y_{t-2}$. Hence: $a_1 = 0.7$ and $a_2 = 0.35$

Form the homogeneous equation: $y_t - 0.7y_{t-1} - 0.35y_{t-2} = 0$

Thus $d = \ + 4a_2 = 1.89$. Given that $d > 0$, the roots will be real and distinct. Form the characteristic equation $\alpha^t - 0.7\alpha^{t-1} - 0.35\alpha^{t-2} = 0$

Divide by α^{t-2} to obtain the characteristic equation: $\alpha^2 - 0.7\alpha - 0.35 = 0$

Compute the two characteristic roots: $\alpha_1 = 1.037$ $\alpha_2 = -0.337$

The homogeneous solution is: $A_1(1.037)^t + A_2(-0.337)^t$.

Section 8

THE METHOD OF UNDETERMINED COEFFICIENTS

The Method of Undetermined Coefficients

Consider the simple first-order equation: $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$

Posit the challenge solution:

$$y_t = b_0 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i}$$

$$b_0 + a_0\varepsilon_t + a_1\varepsilon_{t-1} + a_2\varepsilon_{t-2} + \ldots = a_0 + a_1[b_0 + a_0\varepsilon_{t-1} + a_1\varepsilon_{t-2} + \ ] + \varepsilon_t$$

$$\alpha_0 - 1 = 0$$

$$a_1 - a_1 a_0 = 0$$

$$a_2 - a_1 a_1 = 0$$

$$b_0 - a_0 - a_1 b_0 = 0$$

$$y_t = \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$$

The Method of Undetermined Coefficients II

Consider:

$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \varepsilon_t$ (1.68)

Since we have a second-order equation, we use the challenge solution

$y_t = b_0 + b_1 t + b_2 t^2 + a_0 \varepsilon_t + a_1 \varepsilon_{t-1} + a_2 \varepsilon_{t-2} +$

where b_0, b_1, b_2, and the a_i are the undetermined coefficients. Substituting the challenge solution into (1.68) yields

$$[b_0 + b_1 t + b_2 t^2] + a_0\varepsilon_t + a_1\varepsilon_{t-1} + a_2\varepsilon_{t-2} + \; = a_0 + a_1[b_0 + b_1(t-1) + b_2(t-1)^2$$
$$+ a_0\varepsilon_{t-1} + a_1\varepsilon_{t-2} + a_2\varepsilon_{t-3} + \;] + a_2[b_0 + b_1(t-2) + b_2(t-2)^2$$
$$+ a_0\varepsilon_{t-2} + a_1\varepsilon_{t-3} + a_2\varepsilon_{t-4} + \;] + \varepsilon_t$$

Hence:

$a_0 = 1$

$a_1 = a_1 a_0$ [so that $a_1 = a_1$]

$a_2 = a_1 a_1 + a_2 a_0$ [so that $a_2 = (a_1)^2 + a_2$]

$a_3 = a_1 a_2 + a_2 a_1$ [so that $a_3 = (a_1)^3 + 2a_1 a_2$]

Notice that for any value of $j \geq 2$, the coefficients solve the second-order difference equation $a_j = a_1 a_{j-1} + a_2 a_{j-2}$.

Section 9

- Lag Operators in Higher-Order Systems

LAG OPERATORS

Lag Operators

The lag operator L is defined to be:

$$L^i y_t = y_{t-i}$$

Thus, L^i preceding y_t simply means to lag y_t by i periods.

The lag of a constant is a constant: $Lc = c$.

The distributive law holds for lag operators. We can set:

$$(L^i + L^j)y_t = L^i y_t + L^j y_t = y_{t-i} + y_{t-j}$$

Lag Operators *(cont'd)*

- Lag operators provide a concise notation for writing difference equations. Using lag operators, the p-th order equation

$y_t = a_0 + a_1 y_{t-1} + \ldots + a_p y_{t-p} + \varepsilon_t$ can be written as:

$(1 - a_1 L - a_2 L^2 - \ldots - a_p L^p) y_t = \varepsilon_t$

or more compactly as:

$$A(L) y_t = \varepsilon_t$$

As a second example,

$y_t = a_0 + a_1 y_{t-1} + \ldots + a_p y_{t-p} + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \ldots + \beta_q \varepsilon_{t-q}$ as:

$$A(L) y_t = a_0 + B(L) \varepsilon_t$$

where: $A(L)$ and $B(L)$ are polynomials of orders p and q, respectively.

APPENDIX 1.1: IMAGINARY ROOTS AND DE MOIVRE'S THEOREM

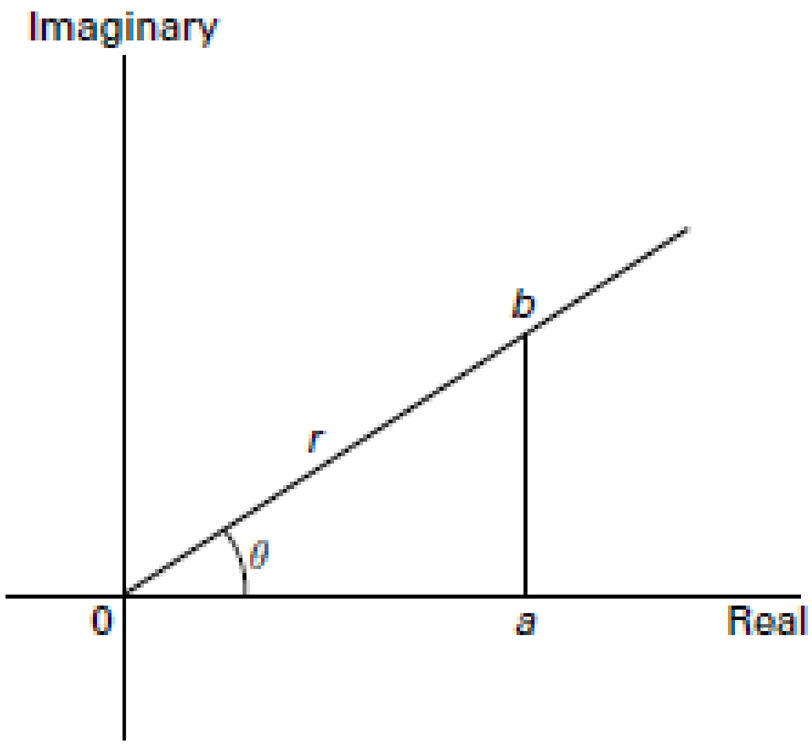

FIGURE A1.1 A Graphical Representation of Complex Numbers