Chapter 1: Difference Equations
Section 1

TIME-SERIES MODELS
The traditional use of time series models was for forecasting.

If we know

\[ y_{t+1} = a_0 + a_1 y_t + \varepsilon_{t+1} \]

then

\[ E_y y_{t+1} = a_0 + a_1 y_t \]

and since

\[ y_{t+2} = a_0 + a_1 y_{t+1} + \varepsilon_{t+2} \]
\[ E_y y_{t+2} = a_0 + a_1 E_y y_{t+1} \]
\[ = a_0 + a_1 (a_0 + a_1 y_t) \]
\[ = a_0 + a_1 a_0 + (a_1)^2 y_t \]
Capturing Dynamic Relationships

• With the advent of modern dynamic economic models, the newer uses of time series models involve
  – Capturing dynamic economic relationships
  – Hypothesis testing
• Developing “stylized facts”
  – In a sense, this reverses the so-called scientific method in that modeling goes from developing models that follow from the data.
The Random Walk Hypothesis

\[ y_{t+1} = y_t + \varepsilon_{t+1} \]

or

\[ \Delta y_{t+1} = \varepsilon_{t+1} \]

where \( y_t \) = the price of a share of stock on day \( t \), and \( \varepsilon_{t+1} \) = a random disturbance term that has an expected value of zero.

Now consider the more general stochastic difference equation

\[ \Delta y_{t+1} = a_0 + a_1 y_t + \varepsilon_{t+1} \]

The random walk hypothesis requires the testable restriction:

\[ a_0 = a_1 = 0. \]
The Unbiased Forward Rate (UFR) hypothesis

Given the UFR hypothesis, the forward/spot exchange rate relationship is:

\[ s_{t+1} = f_t + \varepsilon_{t+1} \]  \hspace{1cm} (1.6)

where \( \varepsilon_{t+1} \) has a mean value of zero from the perspective of time period \( t \).

Consider the regression

\[ s_{t+1} = a_0 + a_1 f_t + \varepsilon_{t+1} \]

The hypothesis requires \( a_0 = 0 \), \( a_1 = 1 \), and that the regression residuals \( \varepsilon_{t+1} \) have a mean value of zero from the perspective of time period \( t \).

The spot and forward markets are said to be in long-run equilibrium when \( \varepsilon_{t+1} = 0 \). Whenever \( s_{t+1} \) turns out to differ from \( f_t \), some sort of adjustment must occur to restore the equilibrium in the subsequent period. Consider the adjustment process

\[ s_{t+2} = s_{t+1} - a[ s_{t+1} - f_t ] + \varepsilon_{s_{t+2}} \quad a > 0 \]  \hspace{1cm} (1.7)

\[ f_{t+1} = f_t + b[ s_{t+1} - f_t ] + \varepsilon_{f_{t+1}} \quad b > 0 \]  \hspace{1cm} (1.8)

where \( \varepsilon_{s_{t+2}} \) and \( \varepsilon_{f_{t+1}} \) both have an expected value of zero.
Trend-Cycle Relationships

- We can think of a time series as being composed of:

\[ y_t = \text{trend} + \text{“cycle”} + \text{noise} \]

- Trend: Permanent
- Cycle: predictable (albeit temporary)
  - (Deviations from trend)
- Noise: unpredictable
FIGURE 1.1 Hypothetical Time Series
Series with decidedly upward trend

Figure 3.1 Real GDP, Consumption and Investment
GDP Volatility?

Figure 3.2 Annualized Growth Rate of Real GDP

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Figure 3.3: Daily Changes in the NYSE US 100 Index: (Jan 4, 2000 - July 16, 2012)
Co-Movements

Figure 3.4 Short- and Long-Term Interest Rates

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Common Trends

Figure 3.5: Daily Exchange Rates (Jan 3, 2000 - April 4, 2013)
Section 2

DIFFERENCE EQUATIONS AND THEIR SOLUTIONS
Consider the function \( y_{t*} = f(t*) \)

\[
\Delta y_{t*+h} \equiv f(t*+h) - f(t*) \\
\equiv y_{t*+h} - y_{t*}
\]

We can then form the first differences:

\[
\Delta y_t = f(t) - f(t-1) \equiv y_t - y_{t-1} \\
\Delta y_{t+1} = f(t+1) - f(t) \equiv y_{t+1} - y_t \\
\Delta y_{t+2} = f(t+2) - f(t+1) \equiv y_{t+2} - y_{t+1}
\]

More generally, for the forcing process \( x_t \) a \( n \)-th order linear process is

\[
y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t
\]
What is a solution?

A **solution** to a difference equation expresses the value of $y_t$ as a function of the elements of the $\{x_t\}$ sequence and $t$ (and possibly some given values of the $\{y_t\}$ sequence called **initial conditions**).

$$y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t$$

The key property of a solution is that it satisfies the difference equation for all permissible values of $t$ and $\{x_t\}$. 
Section 3
• Iteration without an Initial Condition
• Reconciling the Two Iterative Methods
• Nonconvergent Sequences

SOLUTION BY ITERATION
Solution by Iteration

Consider the first-order equation

\[ y_t = a_0 + a_1 y_{t-1} + \varepsilon_t \quad (1.17) \]

Given the value of \( y_0 \), it follows that \( y_1 \) will be given by

\[ y_1 = a_0 + a_1 y_0 + \varepsilon_1 \]

In the same way, \( y_2 \) must be

\[
\begin{align*}
y_2 &= a_0 + a_1 y_1 + \varepsilon_2 \\
    &= a_0 + a_1 [a_0 + a_1 y_0 + \varepsilon_1] + \varepsilon_2 \\
    &= a_0 + a_0 a_1 + (a_1)^2 y_0 + a_1 \varepsilon_1 + \varepsilon_2
\end{align*}
\]

Continuing the process in order to find \( y_3 \), we obtain

\[
\begin{align*}
y_3 &= a_0 + a_1 y_2 + \varepsilon_3 \\
    &= a_0 [1 + a_1 + (a_1)^2] + (a_1)^3 y_0 + a_1^2 \varepsilon_1 + a_1 \varepsilon_2 + \varepsilon_3
\end{align*}
\]
From

\[ y_3 = a_0[1 + a_1 + (a_1)^2] + (a_1)^3 y_0 + a_1^2 \varepsilon_1 + a_1 \varepsilon_2 + \varepsilon_3 \]

you can verify that for all \( t > 0 \), repeated iteration yields

\[
y_t = a_0 \sum_{i=0}^{t-1} a_i^t + a_1^t y_0 + \sum_{i=0}^{t-1} a_i^i \varepsilon_{t-i}
\]

If \( |a_1| < 1 \), in the limit

\[
y_t = a_0 / (1 - a_1) + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}
\]
Backwards Iteration

Iteration from $y_t$ back to $y_0$ yields exactly the formula given by (1.18).

Since $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$, it follows that

$$y_t = a_0 + a_1 [a_0 + a_1 y_{t-2} + \varepsilon_{t-1}] + \varepsilon_t$$

$$= a_0(1 + a_1) + a_1 \varepsilon_{t-1} + \varepsilon_t + a_1^2[a_0 + a_1 y_{t-3} + \varepsilon_{t-2}]$$

If $|a_1| < 1$, in the limit

$$y_t = \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$$
FIGURE 1.2 Convergent and Nonconvergent Sequences
Section 4

• The Solution Methodology
• Generalizing the Method

AN ALTERNATIVE SOLUTION METHODOLOGY
Does it converge? Characteristic Roots

Since the intercept and the $\varepsilon_t$ sequence have nothing to do with the issue of convergence, consider:

$$y_t = a_1 y_{t-1}$$

A solution is

$$y_t = A (a_1)^t$$

Proof:

$$A a_1^t = a_1 A a_1^{t-1}$$

• If $|a_1| < 1$, the $y_t$ converges to zero as $t$ approaches infinity. Convergence is direct if $0 < a_1 < 1$ and oscillatory if $-1 < a_1 < 0$.

• If $|a_1| > 1$, the homogeneous solution is not convergent. If $a_1 > 1$, $y_t$ approaches $\infty$ as $t$ increases. If $a_1 < -1$, the $y_t$ oscillates explosively.

• If $a_1 = 1$, any arbitrary constant $A$ satisfies the homogeneous equation $y_t = y_{t-1}$. If $a_1 = -1$, the system is meta-stable: $= 1$ for even values of $t$ and $-1$ for odd values of $t$. 

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Generalizing the Method

\[ y_t = \sum_{i=1}^{n} a_i y_{t-i} \]

\[ A\alpha^t - a_1 A\alpha^{t-1} - a_2 A\alpha^{t-2} - \ldots - a_n A\alpha^{t-n} = 0 \]

There are \( n \) roots

In the \( n \)th-order case

\[ y_t^h = A_1(\alpha_1)^t + A_2(\alpha_2)^t + \ldots \]

For convergence, all of the roots must be less than unity in absolute value (or inside the unit circle if complex).
In an $n$th-order equation, a necessary condition for all characteristic roots to lie inside the unit circle is

$$\sum_{i=1}^{n} a_i < 1$$

Since the values of the $a_i$ can be positive or negative, a sufficient condition for all characteristic roots to lie inside the unit circle is

$$\sum_{i=1}^{n} |a_i| < 1$$

At least one characteristic root equals unity if

$$\sum_{i=1}^{n} a_i = 1$$

Any sequence that contains one or more characteristic roots that equal unity is called a **unit root** process.

For a third-order equation, the stability conditions can be written as

\[
\begin{align*}
1 - a_1 - a_2 - a_3 & > 0 \\
1 + a_1 - a_2 + a_3 & > 0 \\
1 - a_1a_3 + a_2 - a_3^2 & > 0 \\
3 + a_1 + a_2 - 3a_3 & > 0 \quad \text{or} \quad 3 - a_1 + a_2 + 3a_3 & > 0 \\
\end{align*}
\]

Given that the first three inequalities are satisfied, one of the last conditions is redundant.
The Solution Methodology

**STEP 1:** form the homogeneous equation and find all \( n \) homogeneous solutions;

**STEP 2:** find a particular solution;

**STEP 3:** obtain the general solution as the sum of the particular solution and a linear combination of all homogeneous solutions;

**STEP 4:** eliminate the arbitrary constant(s) by imposing the initial condition(s) on the general solution.
Section 5

- Stability Conditions
- Higher-Order Systems

THE COBWEB MODEL
Setting supply equal to demand:

\[ b + \beta p_{t-1} + \epsilon_t = a - \gamma p_t \]

or

\[ p_t = (-\beta/\gamma)p_{t-1} + (a - b)/\gamma - \epsilon_t/\gamma \]

The homogeneous equation is \( p_t = (-\beta/\gamma)p_{t-1} \).

If the ratio \( \beta/\gamma \) is less than unity, you can iterate (1.39) backward from \( p_t \) to verify that the particular solution for the price is

\[ p_t^p = \frac{a - b}{\gamma + \beta} - \frac{1}{\gamma} \sum_{i=0}^{\infty} (-\beta/\gamma)^i \epsilon_{t-i} \]

Stability requires \( |\beta/\gamma| < 1 \).
FIGURE 1.3  The Cobweb Model
Section 6

- Stability Conditions
- Higher-Order Systems

SOLVING HOMOGENEOUS DIFFERENCE EQUATIONS
Consider the second-order equation

\[ y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0 \]

\[ A\alpha^t - a_1 A\alpha^{t-1} - a_2 A\alpha^{t-2} = 0 \]

If you divide (1.46) by \( A\alpha^{t-2} \), the problem is to find the values of \( \alpha \) that satisfy

\[ \alpha^2 - a_1 \alpha - a_2 = 0 \]

There are two characteristic roots. Hence the homogeneous solution is

\[ A_1(\alpha_1)^t + A_2(\alpha_2)^t \]
THE THREE CASES

CASE 1
If $a_1^2 + 4a_2 > 0$, $d$ is a real number and there will be two distinct real characteristic roots.

CASE 2
If $a_1^2 + 4a_2 = 0$, it follows that $d = 0$ and $a_1 = a_2 = a_1/2$. A homogeneous solution is $a_1/2$. However, when $d = 0$, there is a second homogeneous solution given by $t(a_1/2)^t$.

CASE 3
If $a_1^2 + 4a_2 < 0$, it follows that $d$ is negative so that the characteristic roots are imaginary.
FIGURE 1.4  The Homogeneous Solution of $t(a_1)^f$
FIGURE 1.5  Characterizing the Stability Conditions
FIGURE 1.6  Characteristic Roots and the Unit Circle
WORKSHEET 1.1: SECOND-ORDER EQUATIONS

**Example 1**: \( y_t = 0.2y_{t-1} + 0.35y_{t-2} \). Hence: \( a_1 = 0.2 \) and \( a_2 = 0.35 \)

Form the homogeneous equation: \( y_t - 0.2y_{t-1} - 0.35y_{t-2} = 0 \)

\[ d = +4a_2 \text{ so that } d = 1.44. \text{ Given that } d > 0, \text{ the roots will be real and distinct.} \]

Substitute \( y_t = \alpha \) into the homogenous equation to obtain: \( \alpha^2 - 0.2\alpha - 0.35 = 0 \)

Divide by \( \alpha^{-2} \) to obtain the characteristic equation: \( \alpha^2 - 0.2\alpha - 0.35 = 0 \)

Compute the two characteristic roots: \( \alpha_1 = 0.7 \quad \alpha_2 = -0.5 \)

The homogeneous solution is: \( A_1(0.7)t + A_2(-0.5)t \).

**Example 2**: \( y_t = 0.7y_{t-1} + 0.35y_{t-2} \). Hence: \( a_1 = 0.7 \) and \( a_2 = 0.35 \)

Form the homogeneous equation: \( y_t - 0.7y_{t-1} - 0.35y_{t-2} = 0 \)

Thus \( d = +4a_2 = 1.89. \text{ Given that } d > 0, \text{ the roots will be real and distinct.} \)

Form the characteristic equation \( \alpha^2 - 0.7\alpha - 0.35 = 0 \)

Divide by \( \alpha^{-2} \) to obtain the characteristic equation: \( \alpha^2 - 0.7\alpha - 0.35 = 0 \)

Compute the two characteristic roots: \( \alpha_1 = 1.037 \quad \alpha_2 = -0.337 \)

The homogeneous solution is: \( A_1(1.037)t + A_2(-0.337)t \).
Section 8

THE METHOD OF UNDETERMINED COEFFICIENTS
The Method of Undetermined Coefficients

Consider the simple first-order equation: \( y_t = a_0 + a_1 y_{t-1} + \varepsilon_t \)

Posit the challenge solution:

\[
y_t = b_0 + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i}
\]

\[
b_0 + a_0 \varepsilon_t + a_1 \varepsilon_{t-1} + a_2 \varepsilon_{t-2} + \ldots = a_0 + a_1 [b_0 + a_0 \varepsilon_{t-1} + a_1 \varepsilon_{t-2} + \ldots] + \varepsilon_t
\]

\[
\alpha_0 - 1 = 0
\]
\[
a_1 - a_1 a_0 = 0
\]
\[
a_2 - a_1 a_1 = 0
\]
\[
b_0 - a_0 - a_1 b_0 = 0
\]

\[
y_t = \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}
\]
The Method of Undetermined Coefficients II

Consider:
\[ y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \varepsilon_t \]  \hspace{1cm} (1.68)

Since we have a second-order equation, we use the challenge solution
\[ y_t = b_0 + b_1 t + b_2 t^2 + a_0 \varepsilon_t + a_1 \varepsilon_{t-1} + a_2 \varepsilon_{t-2} + \]

where \( b_0, b_1, b_2 \), and the \( a_i \) are the undetermined coefficients. Substituting the challenge solution into (1.68) yields
\[
[b_0 + b_1 t + b_2 t^2] + a_0 \varepsilon_t + a_1 \varepsilon_{t-1} + a_2 \varepsilon_{t-2} + = a_0 + a_1 [b_0 + b_1 (t - 1) + b_2 (t - 1)^2 \\
+ a_0 \varepsilon_{t-1} + a_1 \varepsilon_{t-2} + a_2 \varepsilon_{t-3} + ] + a_2 [b_0 + b_1 (t - 2) + b_2 (t - 2)^2 \\
+ a_0 \varepsilon_{t-2} + a_1 \varepsilon_{t-3} + a_2 \varepsilon_{t-4} + ] + \varepsilon_t
\]

Hence:
\[
\begin{align*}
a_0 &= 1 \\
a_1 &= a_1 a_0 \\
a_2 &= a_1 a_1 + a_2 a_0 \\
a_3 &= a_1 a_2 + a_2 a_1
\end{align*}
\]

[so that \( a_1 = a_1 \)]

[so that \( a_2 = (a_1)^2 + a_2 \)]

[so that \( a_3 = (a_1)^3 + 2a_1 a_2 \)]

Notice that for any value of \( j \geq 2 \), the coefficients solve the second-order difference equation \( a_j = a_1 a_{j-1} + a_2 a_{j-2} \).
Section 9

- Lag Operators in Higher-Order Systems

LAG OPERATORS
Lag Operators

The lag operator $L$ is defined to be:

$$L^i y_t = y_{t-i}$$

Thus, $L^i$ preceding $y_t$ simply means to lag $y_t$ by $i$ periods.

The lag of a constant is a constant: $Lc = c$.

The distributive law holds for lag operators. We can set:

$$(L^i + L^j)y_t = L^i y_t + L^j y_t = y_{t-i} + y_{t-j}$$
Lag Operators (cont’d)

- Lag operators provide a concise notation for writing difference equations. Using lag operators, the $p$-th order equation

\[ y_t = a_0 + a_1 y_{t-1} + \ldots + a_p y_{t-p} + \varepsilon_t \]

can be written as:

\[(1 - a_1 L - a_2 L^2 - \ldots - a_p L^p) y_t = \varepsilon_t \]

or more compactly as:

\[ A(L)y_t = \varepsilon_t \]

As a second example,

\[ y_t = a_0 + a_1 y_{t-1} + \ldots + a_p y_{t-p} + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \ldots + \beta_q \varepsilon_{t-q} \]

as:

\[ A(L)y_t = a_0 + B(L)\varepsilon_t \]

where: $A(L)$ and $B(L)$ are polynomials of orders $p$ and $q$, respectively.
APPENDIX 1.1: IMAGINARY ROOTS AND DE MOIVRE’S THEOREM

FIGURE A1.1 A Graphical Representation of Complex Numbers