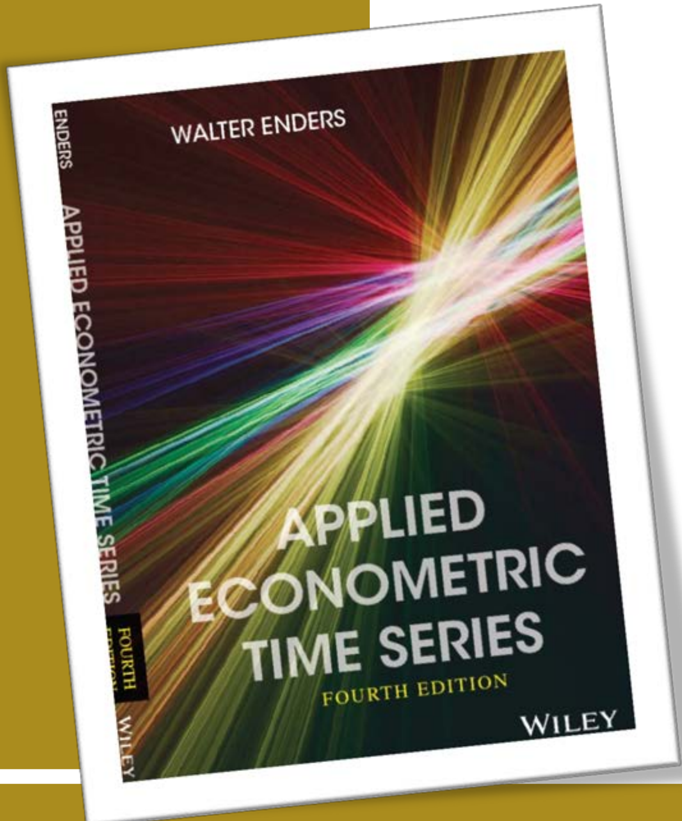

Applied Econometric Time Series 4th ed. Walter Enders

Chapter 6

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Example of Cointegration and Money Demand

- In logarithms, an econometric specification for such an equation can be written as:

$$m_t = b_0 + b_1 p_t + b_2 y_t + b_3 r_t + e_t$$

where: m_t = demand for money

p_t = price level

y_t = real income

r_t = interest rate

e_t = *stationary* disturbance term

b_i = parameters to be estimated

Other Examples

- *Consumption function theory.*
- *Unbiased forward rate hypothesis.*
- *Commodity market arbitrage and purchasing power parity.*
- The formal analysis begins by considering a set of economic variables in long-run equilibrium when

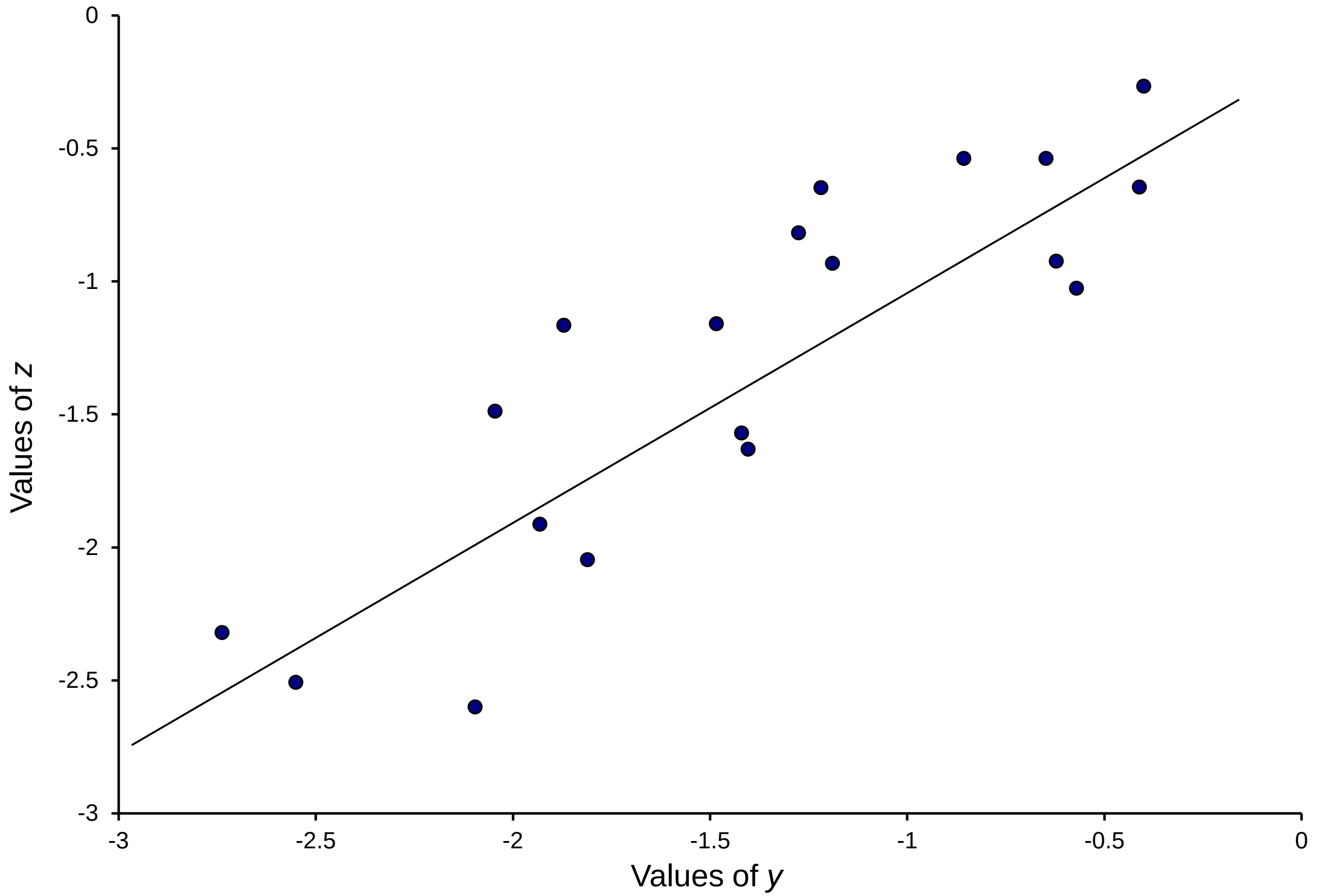
$$\beta_1 x_{1t} + \beta_2 x_{2t} + \ldots + \beta_n x_{nt} = 0$$

- Letting β and x_t denote the vectors $(\beta_1, \beta_2, \ldots, \beta_n)$ and $(x_{1t}, x_{2t}, \ldots, x_{nt})'$, the system is in long-run equilibrium when $bx_t = 0$. The deviation from long-run equilibrium—called the **equilibrium error**—is e_t, so that
- $e_t = \beta x_t$

Generalization

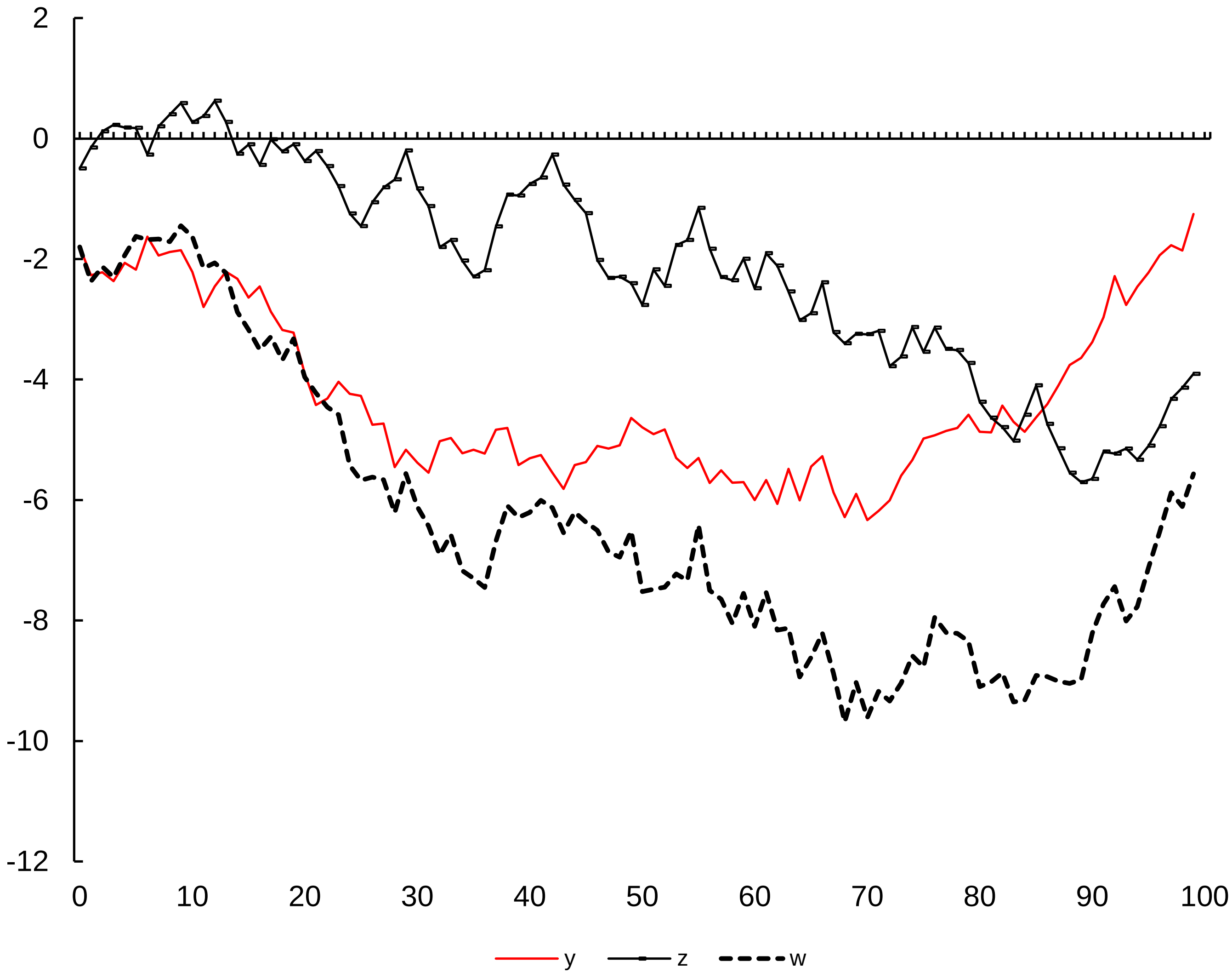
- Letting β and x_t denote the vectors $(\beta_1, \beta_2, ..., \beta_n)$ and $(x_{1t}, x_{2t}, ..., x_{nt})$, the system is in long-run equilibrium when $\beta x_t' = 0$. The deviation from long-run equilibrium--called the **equilibrium error**--is e_t, so that:
- $$e_t = \beta x'_t$$
- If the equilibrium is meaningful, it must be the case that the equilibrium error process is stationary.

Figure 6.1: Scatter Plot of Cointegrated Variables



The scatter plot was drawn using the {*y*} and {*z*} sequences from Case 1 of Worksheet 6.1. Since both series decline over time, there appears to be a positive relationship between the two. The equilibrium regression line is shown.

Figure 6.2: Three Cointegrated Series



Three important points

- 1. Cointegration refers to a **linear** combination of non-stationary variables.
  - If $(\beta_1, \beta_2, \ldots, \beta_n)$ is a cointegrating vector, then for any non-zero value of λ, $(\lambda\beta_1, \lambda\beta_2, \ldots, \lambda\beta_n)$ is also a cointegrating vector.
  - Typically, one of the variables is used to *normalize* the cointegrating vector by fixing its coefficient at unity.
    - To normalize the cointegrating vector with respect to x_{1t}, simply select $\lambda = 1/\beta_1$.
- 2. The equation must be balanced in that the order of integration of the two sides must be equal
- 3. If x_t has m components, there may be as many as *m*-1 linearly independent cointegrating vectors

Example of Multiple Cointegrating Vectors

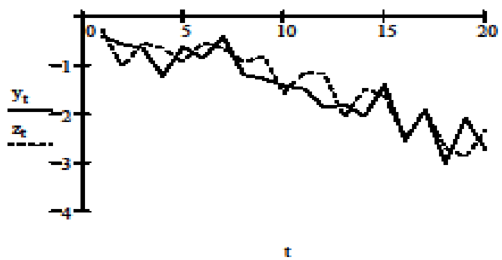
- Let the money supply rule be:
- $m_t = \gamma_0 - \gamma_1(y_t + p_t) + e_{1t}$ (1.3)
- $= \gamma_0 - \gamma_1 y_t - \gamma_1 p_t + e_{1t}$
- where: $\{e_{1t}\}$ is a stationary error in the money supply feedback rule.
- Given the money demand function in (1.1), there are two cointegrating vectors for the money supply, price level, real income, and the interest rate. Let β be the (5 x 2) matrix:

$$\beta = \begin{bmatrix} 1 & -\beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ 1 & -\gamma_0 & \gamma_1 & \gamma_1 & 0 \end{bmatrix}$$

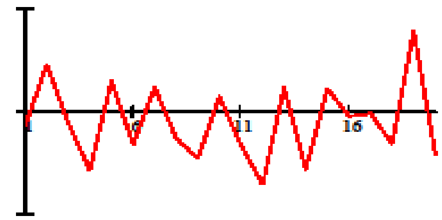
WORKSHEET 6.1: Illustrating Cointegrated Systems

CASE 1: The series $\{\mu_t\}$ is a random walk process and $\{\varepsilon_{yt}\}$ and $\{\varepsilon_{zt}\}$ are white noise. Hence, the $\{y_t\}$ and $\{z_t\}$ sequences are both random walk plus noise processes. Although each is nonstationary, the two sequences have the same stochastic trend; hence they are cointegrated such that the linear combination $(y_t - z_t)$ is stationary. The equilibrium error term $(\varepsilon_{yt} - \varepsilon_{zt})$ is an $I(0)$ process.

$y_t = \mu_t + \varepsilon_{yt}$ $\quad z_t = \mu_t + \varepsilon_{zt}$

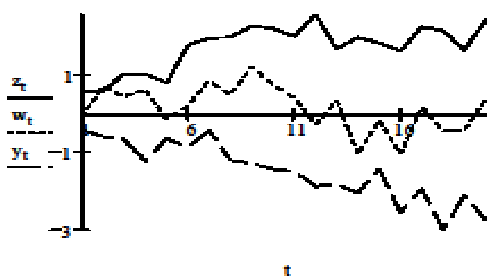


The equilibrium error: $y_t - z_t$

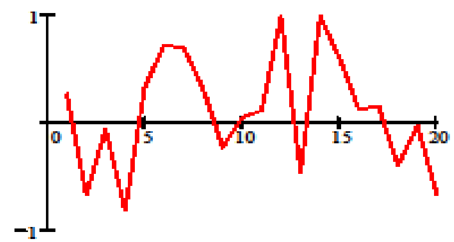


CASE 2: All three sequences are random walk plus noise processes. As constructed no two are cointegrated. However, the linear combination $(y_t + z_t - w_t)$ is stationary; hence, the three variables are cointegrated. The equilibrium error is an $I(0)$ process.

$y_t = \mu_{yt} + \varepsilon_{yt}$, $\quad z_t = \mu_{zt} + \varepsilon_{zt}$, $\quad w_t = \mu_{wt} + \varepsilon_{wt}$



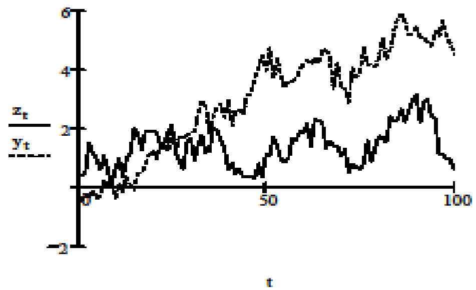
The equilibrium error: $y_t + z_t - w_t$



Worksheet 6.2: Non-integrated Variables

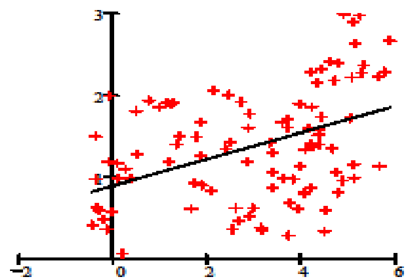
The $\{y_t\}$ and $\{z_t\}$ sequences are constructed to independent random walk plus noise processes. There is **no** cointegrating relationship between the two variables. As shown in graph (a), both seem to meander without any tendency to come together. Graph (b) shows the scatter plot of the two sequences and the regression line $z_t = \beta_0 + \beta_1 y_t$. However, this regression line is spurious. As shown in graph (c), the regression residuals are nonstationary.

$y_t = \mu_{yt} + \varepsilon_{yt}$ $\qquad$ $z_t = \mu_{zt} + \varepsilon_{zt}$



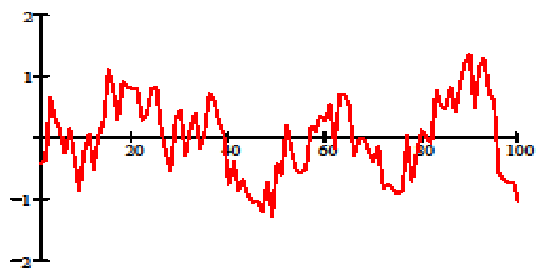
(a)

Regression of z_t on y_t



(b)

Regression Residuals



(c)

COINTEGRATION AND COMMON TRENDS

- $y_t = \mu_{yt} + e_{yt}$
- $z_t = \mu_{zt} + e_{zt}$
  - *where* μ_{it} = a random walk process representing the trend in variable i
  - e_{it} = the stationary (irregular) component of variable i
- If $\{y_t\}$ and $\{z_t\}$ are cointegrated of order (1,1), there must be nonzero values of β_1 and β_2 for which the linear combination $\beta_1 y_t + \beta_2 z_t$ is stationary. Consider the sum

$$\beta_1 y_t + \beta_2 z_t = \beta_1(\mu_{yt} + e_{yt}) + \beta_2(\mu_{zt} + e_{zt})$$

$$= (\beta_1\mu_{yt} + \beta_2\mu_{zt}) + (\beta_1 e_{yt} + \beta_2 e_{zt}) \quad (6.6)$$

For $\beta_1 y_t + \beta_2 z_t$ to be stationary, the term $(\beta_1\mu_{yt} + \beta_2\mu_{zt})$ must vanish.

Granger Representation Theorem

- In an error-correction model, the short-term dynamics of the variables in the system are influenced by the deviation from equilibrium.

$$\Delta r_{St} = \alpha_S(r_{Lt-1} - \beta r_{St-1}) + \varepsilon_{St} \qquad \alpha_S > 0$$

$$\Delta r_{Lt} = -\alpha_L(r_{Lt-1} - \beta r_{St-1}) + \varepsilon_{Lt} \qquad \alpha_L > 0$$

This finding illustrates the **Granger representation theorem** stating that for any set of $I(1)$ variables, error correction and cointegration are equivalent representations.

The Engle-Granger Methodology

Step 1: Pretest the variables for their order of integration.

Step 2: Estimate the long-run equilibrium relationship.

If the results of Step 1 indicate that both $\{y_t\}$ and $\{z_t\}$ are $I(1)$, the next step is to estimate the long-run equilibrium relationship in the form:

$$y_t = \beta_0 + \beta_1 z_t + e_t$$

Consider the autoregression of the residuals:

$$\Delta\hat{e}_t = a_1\hat{e}_{t-1} + \sum_{i=1}^{n} a_{i+1}\Delta\hat{e}_{t-i} + \varepsilon_t$$

Test $a_1 = 0$?

Step 3: Estimate the error-correction model

The Error Correction Model

$$\Delta y_t = \alpha_1 + \alpha_y\left[y_{t-1} - \beta_1 z_{t-1}\right] + \sum_{i=1} a_{11}(i)\Delta y_{t-i} + \sum_{i=1} a_{12}(i)\Delta z_{t-i} + \varepsilon_{yt}$$

$$\Delta z_t = \alpha_2 + \alpha_z\left[y_{t-1} - \beta_1 z_{t-1}\right] + \sum_{i=1} a_{21}(i)\Delta y_{t-i} + \sum_{i=1} a_{22}(i)\Delta z_{t-i} + \varepsilon_{zt}$$

Instead of a cross-equation restriction, use

$$\Delta y_t = \alpha_1 + \alpha_y \hat{e}_{t-1} + \sum_{i=1} a_{11}(i)\Delta y_{t-i} + \sum_{i=1} a_{12}(i)\Delta z_{t-i} + \varepsilon_{yt}$$

$$\Delta z_t = \alpha_2 + \alpha_z \hat{e}_{t-1} + \sum_{i=1} a_{21}(i)\Delta y_{t-i} + \sum_{i=1} a_{22}(i)\Delta z_{t-i} + \varepsilon_{zt}$$

Speed of adjustment coefficients

The *speed of adjustment* coefficients α_y and α_z are of particular interest in that they have important implications for the dynamics of the system.

Direct convergence necessitates that be negative and α_z be positive. If we focus on (6.36) it is clear that for any given value of the deviation from long-run equilibrium, a large value of α_z is associated with a large value of Δz_t.

If one of these coefficients is (say α_y) is zero,the $\{z_t\}$ sequence does all of the correction to eliminate any deviation from long-run equilibrium. Since $\{y_t\}$ does not do any of the error-correcting, $\{y_t\}$ is said to be **weakly exogenous**.

Problems with the EG-Method

1. In practice, it is possible to find that one regression indicates the variables are cointegrated whereas reversing the order indicates no cointegration. This is a very undesirable feature of the procedure since the test for cointegration should be invariant to the choice of the variable selected for normalization. The problem is obviously compounded using three or more variables since any of the variables can be selected as the left-hand-side variable.

- 2. Moreover, in tests using three or more variables, we know that there may be more than one cointegrating vector. The method has no systematic procedure for the separate estimation of the multiple cointegrating vectors.

- 3. Another serious defect of the Engle-Granger procedure is that it relies on a **two-step** estimator.

Johansen Methodology

Reconsider the n-variable first-order VAR given by (6.3): $x_t = A_1x_{t-1} + \varepsilon_t$. Subtract x_{t-1} from each side to obtain:

$$\begin{aligned}\Delta x_t &= A_1 x_{t-1} - x_{t-1} + \varepsilon_t \\ &= (A_1 - I)x_{t-1} + \varepsilon_t \\ &= \pi x_{t-1} + \varepsilon_t\end{aligned}$$

The rank of $(A_1 – I)$ equals the number of cointegrating vectors.

If $(A_1 – I)$ consists of all zeroes—so that rank(π) = 0—all of the $\{x_{it}\}$ sequences are unit root processes.

If $(A_1 – I)$ is of full rank—so that rank(π) = n—each of the $\{x_{it}\}$ sequences converges to a point.

The process can be modified to include a drift and seasonal dummy variables.

Consider $\Delta x_t = \pi^* x_{t-1}$:

$$\pi^* = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1n} & \pi_{10} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2n} & \pi_{20} \\ . & . & \cdots & . & . \\ \pi_{n1} & \pi_{n}2 & \cdots & \pi_{nn} & \pi_{n0} \end{bmatrix}$$

If rank $\pi^* = 2$,

$$\pi_{11}x_{1t} + \pi_{12}x_{2t} + \pi_{13}x_{3t} + \ldots + \pi_{1n}x_{1n} + \pi_{10} = 0$$
$$\pi_{21}x_{1t} + \pi_{22}x_{2t} + \pi_{23}x_{3t} + \ldots + \pi_{2n}x_{1n} + \pi_{20} = 0$$

Note: Adding a column of constants still means that rank(π^*) cannot exceed n

The number of distinct cointegrating vectors can be obtained by checking the significance of the characteristic roots of π. We know that the rank of a matrix is equal to the number of its characteristic roots that differ from zero. Suppose we obtained the matrix π and ordered the *n* characteristic roots such that $\lambda_1 > \lambda_2 > ... > \lambda_n$. If the variables in x_t are *not* cointegrated, the rank of π is zero and all of these characteristic roots will equal zero. Since ln(1) = 0, each of the expressions $\ln(1 - \lambda_i)$ will equal zero if the variables are not cointegrated. Similarly, if the rank of π is to unity, the first expression $\ln(1 - \lambda_1)$ will be negative and all the other expressions are such that $\ln(1 - \lambda_2) = \ln(1 - \lambda_3) = ... = \ln(1 - \lambda_n) = 0$.

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^{n} \ln(1-\hat{\lambda}_i)$$

The null hypothesis that the number of distinct cointegrating vectors is less than or equal to *r* against a general alternative. From the previous discussion, it should be clear that λ_{trace} equals zero when all $\lambda_i = 0$.

$$\lambda_{\max}(r, r+1) = -T \ln(1-\hat{\lambda}_{r+1})$$

The null that the number of cointegrating vectors is *r* against the alternative of r+1 cointegrating vectors. Again, if the estimated value of the characteristic root is close to zero, $\lambda_{\max}$ will be small.

Null Hypothesis	Alternative Hypothesis		95% Critical Value	90% Critical Value
λ_{trace} tests:		λ_{trace} value		
$r = 0$	$r > 0$	44.94926	29.68	26.79
$r <= 1$	$r > 1$	14.80894	15.41	13.33
$r <= 2$	$r > 2$	3.60231	3.76	2.69
λ_{max} tests:		λ_{max} value		
$r = 0$	$r = 1$	30.14032	20.97	18.60
$r = 1$	$r = 2$	11.2066	14.07	12.07
$r = 2$	$r = 3$	3.60231	3.76	2.69

In order to test other restrictions on the cointegrating vector, Johansen defines the two matrices α and β both of dimension (n x r) where r is the rank of π. The properties of α and β are such that:

$$\pi = \alpha \beta'$$

In essence, we can normalize to obtain $\alpha \beta'$

Hypothesis Testing

$$T\sum_{i=r+1}^{n}[\ln(1-\lambda_i^*)-\ln(1-\lambda_i)]$$

Asymptotically, the statistic has a χ^2 distribution with (n - r) degrees of freedom.

The value of this statistic should be zero if the restriction is not binding.

Lag Length and Causality Tests

$$\Delta x_t = \pi x_{t-1} + \sum_{i=1}^{p-1} \pi_i \Delta x_{t-i} + \varepsilon_t$$

Estimate the models with p and $p - 1$ lags. Let c denote the maximum number of regressors contained in the longest equation. The test statistic

$$(T - c)(\log|\Sigma_r| - \log|\Sigma_u|)$$

can be compared to a χ^2 distribution with degrees of freedom equal to the number of restrictions in the system.

Alternatively, you can use the multivariate AIC or SBC to determine the lag length.
If you want to test the lag lengths for a single equation, an F-test is appropriate.

To difference or not to difference?

Difference

- Tests lose power if you do not difference: you estimate n^2 more parameters (one extra lag of each variable in each equation).
- If you use first differences, you can use the standard F distribution to test for Granger causality.
- When the VAR has $I(1)$ variables, the impulse responses at long forecast horizons are inconsistent estimates of the true responses. Since the impulse responses need not decay, any imprecision in the coefficient estimates will have a permanent effect on the impulse responses.

Do not difference

- If the system contains a cointegrating relationship, the system in differences is misspecified since it excludes the long-run equilibrium relationships among the variables that are contained in πx_{t-1}.
  - All of the coefficient estimates, t-tests, F-tests, tests of cross-equation restrictions, impulse responses and variance decompositions are not representative of the true process.

Restrictions on the cointegrating vectors

Testing coefficient restrictions: As in the previous section, once you select the number of cointegrating vectors, you can test restrictions on the resulting values of β and/or α. Suppose you want to test the restriction that the intercept is zero. From the menu, you select *Restrictions on subsets of β*.

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \\ \Phi_{31} \end{bmatrix}$$

Instead, suppose you want to test the **three** restrictions: $\beta_1 = \beta_2$, $\beta_1 = -\beta_3$, and $\beta_3 = 0$ (so that the normalized cointegrating vector has the form $y_t + z_t - w_t = 0$). In matrix form, the

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} [\Phi_{11}]$$

Linear vs Threshold Cointegration

- In the simplest case, the two-step methodology entails using OLS to estimate the long-run equilibrium relationship as:

$$x_{1t} = \beta_0 + \beta_2 x_{2t} + \beta_3 x_{3t} + \dots + \beta_n x_{nt} + e_t$$

- where: x_{it} are the individual $I(1)$ components of x_t, β_i are the estimated parameters, and e_t is the disturbance term which may be serially correlated.
- The second-step focuses on the OLS estimate of ρ in the regression equation:
- $\Delta e_t = \rho e_{t\text{-}1} + \varepsilon_t$

The TAR Specification

Let the error process have the form

$$\Delta e_t = I_t \rho_1 e_{t-1} + (1 - I_t)\rho_2 e_{t-1} + \varepsilon_t$$

where: I_t is the Heaviside indicator function such that:

$$I_t = \begin{cases} 1 & \text{if } e_{t-1} \geq \tau \\ 0 & \text{if } e_{t-1} < \tau \end{cases}$$

The Momentum Specification

$$I_t = \begin{cases} 1 & \text{if } \Delta e_{t-1} \geq 0 \\ 0 & \text{if } \Delta e_{t-1} < 0 \end{cases}$$

TABLE 7: Estimates of the Interest Rate Differential

From Enders and Siklos (JBES)

	Engle-Granger	Threshold	Momentum	Momentum-Consistent
$\rho_1{}^{a}$	-0.068 (-2.858)	-0.085 (-2.522)	-0.021 (-0.628)	-0.020 (-0.680)
$\rho_2{}^{a}$	NA	-0.020 (-1.582)	-0.117 (-3.526)	-0.141 (-3.842)
$\gamma_1{}^{a}$	0.188 (-2.782)	0.190 (2.787)	0.183 (2.730)	0.186 (2.790)
$\gamma_2{}^{a}$	-0.149 (-2.197)	-0.147 (-2.153)	-0.161 (-2.376)	-0.155 (-2.312)
AIC ^{b}	11.74	13.24	9.285	7.022
Φ^{c}	NA	4.32	6.363	7.548
$\rho_1 = \rho_2{}^{d}$	NA	0.495 (0.482)	4.418 (0.037)	6.698 (0.010)
$Q(4)^{e}$ Q(8) Q(12)	0.65 0.60 0.75	0.64 0.58 0.73	0.64 0.52 0.68	0.48 0.51 0.70

$$\Delta x_{it} = \rho_{1.i} I_t e_{t\text{-}1} + \rho_{2.i}(1 - I_t)e_{t\text{-}1} + \ldots + v_{it}$$

where: $\rho_{1.i}$ and $\rho_{2.i}$ are the speed of adjustment coefficients of Δx_{it}.

10. Error-Correction and ADL Tests

$$\Delta y_t = \alpha_1(y_{t-1} - \beta z_{t-1}) + e_{1t}$$
$$\Delta z_t = \alpha_2(y_{t-1} - \beta z_{t-1}) + e_{2t}$$

where: $e_{1t} = \rho e_{2t} + v_t$

As such, we can always write

$$\Delta y_t = \alpha(y_{t-1} - \beta z_{t-1}) + \rho \Delta z_t + v_t \qquad (6.67)$$

The general problem is that Δz_t will be correlated with the error term v_t so that there is a simultaneity problem. However, if z_t is weakly exogenous and causally prior to ye_t we can estimate (6.67)

The ADL Test

$$\Delta y_t = \alpha_1 y_{t-1} - \alpha_1 \beta z_{t-1} + \rho \Delta z_t + v_t$$

Table F uses the work of Ericsson and MacKinnon (2002) to calculate the appropriate critical values necessary to determine whether $\beta_1 < 0$.

Given that the variables are cointegrated:

If Δz_t is unaffected by innovations in Δy_t, it is appropriate to conduct inference on (6.69) using a standard t-tests and F-tests.