On the Use of the Flexible Fourier Form in Unit Roots Tests, Endogenous Breaks, and Parameter Instability

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Abstract

The possibility that a series can contain an unknown number of smooth breaks raises two distinct problems. First, even if the breaks are sharp, the number of breaks and the break dates themselves are generally unknown and need to be estimated along with the other parameters of the model. Second, even if the number of breaks is known, the possibility of a smooth break means that the functional form of the break is unknown to the researcher. A misspecification of the functional form of the breaks may be as problematic as ignoring the breaks altogether. Moreover, ever if a series contains no breaks, it may be subject to other nonlinearities or parameter instabilities. We summarize a number of papers that use a variant of Gallant's (1981) Flexible Fourier Form to control for the unknown number and form of the breaks. The paper details and illustrates several unit root tests, stationarity tests, and tests for parameter instability that are based on a Fourier approximation.

Keywords: Structural breaks, nonlinear models, Fourier approximation.

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1. Introduction

Although nonlinear models have proven to be especially helpful in capturing the types of dynamics exhibited by economic time-series data, the key disadvantage limiting their widespread use is the problem of selecting the proper functional form. Once the applied econometrician leaves the realm of standard linear models, there is a plethora of possible choices for the most appropriate specification to use. As detailed in Terasvirta, Tjostheim and Granger, (2010), the econometrician can select among numerous alternatives including ESTAR, LSTAR, threshold, ANN, Markov switching, random coefficient, GAR, bilinear, and MARS models, just to name a few.[1] As documented in Patterson, and Ashley (2000), even a battery of various Lagrange multiplier (LM) tests and general specification tests cannot pinpoint the precise form of the nonlinearity. It turns out that this expanded choice for the functional form is not always desirable. After all, a misspecification of the functional form may be as problematic as ignoring the nonlinearity altogether.

We synthesize a number of recent papers that use a Fourier series approximation to mitigate the problem of selecting the most appropriate functional form. By way of background, note that under very weak conditions, a Fourier approximation is able to represent any absolutely integrable function to any degree of accuracy. If $d(t)$ is a discrete deterministic function of time (t) over the interval $t = 1, \ldots, T$, it is possible to use a Fourier approximation to capture the behavior of $d(t)$ as:

$$d(t) = \sum_{k=1}^{n} a_k \sin(2\pi kt/T) + \sum_{k=1}^{n} b_k \cos(2\pi kt/T) + e(n) \tag{1}$$

where n is the number of frequencies used in the approximation, the a_k and b_k ($i = 1, \ldots, n$) are parameters, and $e(n)$ is the approximation error. The notation is intended to

[1] The abbreviations stand for exponential smooth transition autoregressive, exponential smooth transition autoregressive, artificial neural network, generalized autoregressive, multiple adaptive regressive splines, respectively.

indicate that the approximation error is a function of the number of frequencies included in equation (1).

There are several notable features of the approximation:

- As n approaches $T/2$, the approximation error continually declines; when $n = T/2$ the approximation error is zero. Intuitively, increasing n increases the number of terms in the approximation; for $n = T/2$, the approximation contains T parameters (the k values of a_k plus the k values of b_k) and is capable of passing through every point of $d(t)$.
- A Fourier series approximation is an orthogonal basis that fully spans the domain of the series in question. This property of the approximation is especially useful for testing purposes since every term in the approximation is uncorrelated with every other term.
- The sine and cosine functions have a maximum value of $+1$ and a minimum value of -1. Hence, unlike some other approximations (such as a Taylor series approximation using powers of *time*), the Fourier approximation is always bounded.
- Many expansions, such as a Taylor series expansion, are necessarily evaluated at a point. Typically, in econometric practice, the point at which the evaluation necessarily occurs is the midpoint of the series in question. However, a Fourier series approximation is a global, not a local, approximation and does not need to be evaluated at a particular point. This is especially useful for models of nonlinear adjustment since the behavior of the series near its midpoint can be quite different from that in its upper and/or lower portions.
- The Fourier approximation can be considered a variant of Gallant's (1981) Flexible Fourier Form (FFF). In essence, the approximation is taken with respect to *time* rather than with respect to a cross-section variable.

- In econometric practice, it is not possible to include all $n = T/2$ frequencies since the resultant estimation would contain no degrees of freedom. In essence, the use of the Fourier approximation transforms the usual problem of selecting the proper functional form to one of selecting the most appropriate frequencies to include in the approximation.

The paper is organized as follows. Becker, Enders and Lee (2006) and Enders and Lee (2011) show that a small number of low frequency components of a Fourier approximation can capture the behavior of a wide variety of structural breaks. Section 2 demonstrates this result and Section 3 shows how to incorporate a Fourier approximation into a unit root test that allows for endogenous structural change. Structural breaks shift the spectral density function toward zero so that it seems reasonable to control for breaks using the low frequency components of a Fourier approximation. It is shown that the Fourier tests have good size and power properties. Section 4 relies on Becker, Enders, and Hurn (2006) and on Enders and Holt (2012) to show how a Fourier approximation can be used to model structural change. It is also the case that parameter instability and other forms of nonlinearity should manifest themselves at the higher end of the spectrum. Section 5 discusses the work of Ludlow and Enders (2000) and Becker, Enders, and Hurn (2004) to show how a Fourier approximation can detect other types of nonlinearity. Section 6 contains our conclusions.

Throughout, the appropriate methodology is illustrated using a number of commodity prices, with a particular emphasis on crude oil prices. In the 1970s, and in recent periods, the real prices of many commodities have experienced sustained run-ups. As such, they are good candidates to examine when looking for series that are subject to possible unit roots, multiple breaks at unknown break dates, and unknown forms of nonlinearity.

Section 2: The Fourier Approximation and Structural Breaks

Most time-series models incorporating structural breaks use dummy variables to capture permanent changes in the level or slope of a series. The implicit assumption is that breaks are sharp in that they occur at a particular point in time and that their effects are felt instantaneously. Nevertheless, a growing literature recognizes that structural change can occur gradually. For example, Perron (1989) modeled the effect of the 1973 oil price shock on the trend growth of the U.S. economy as a single sharp break. However, during 1973, OPEC increased posted prices by 5.7% on April 1, 11.9% on June 1, 17% on October 16, and declared a complete export embargo on October 20. Even if these price jumps are best modeled as being sharp, the effects are likely to be gradual as it took time for the price increases to manifest themselves in output reductions. Moreover, a number of studies suggest that the reduction in trend growth actually began sometime in the late 1960s or very early 1970s (see, for instance, the Symposium on the Slowdown in Productivity Growth in the Journal of Economic Perspectives, 1988). Clearly, any slowdown in R&D activity and in the growth rate of skilled labor would be indicative of a gradual decline in trend GDP. In addition, as in Basu, Fernald and Shapiro (2001), the literature concerning the resumption of high productivity growth in the 1990s suggests the presence of a second smooth break in trend growth. Moreover, Enders and Holt (2012) document that the recent run-ups in petroleum (and other commodity) prices are best modeled as sustained increases rather than sharp breaks. The point is that a model for U.S. GDP and/or petroleum prices allowing for an unknown number of possibly smooth breaks is likely to be superior to a model that contains only a single sharp break.

The possibility that a series can contain an unknown number of smooth breaks raises two distinct problems. First, even if the breaks are sharp, the number of breaks and the break dates themselves are generally unknown and need to be estimated along with the other parameters of the model. Second, even if the number of breaks is known, the possibility of a smooth break means that the functional form of the break is unknown to the researcher. Of course, in applied work both problems can exist simultaneously.

Since structural breaks shift the spectral density function towards zero, the low frequency components of a Fourier approximation can often capture the behavior of a series containing multiple structural breaks. Figures 1 and 2 compare the ability of a Fourier approximation with a small number of frequency components to that of the well-known Bai-Perron (1998) test. Since the Bai-Perron (1998) procedure estimates all breaks as sharp, it is interesting to compare how it performs relative to a Fourier approximation.

The solid lines in the six panels of Figure 1 show the six nonlinear breaks used in Enders and Lee (2012). Specifically, the following equations were used to create the ESTAR and LSTAR breaks shown in the Figure:

	Type of Break	Formula	Value of γ
Panel 1	LSTAR Break at $T/2$	$d(t)=3/(1+\exp(\gamma(t-T/2)))$	$\gamma = 0.05$
Panel 2	"LSTAR Break at $3T/4$	$d(t)=3/(1+\exp(\gamma(t-3T/4)))$	$\gamma = 0.05$
Panel 3	ESTAR Break at $3T/4$	$d(t)=3(1-\exp(-\gamma(t-3T/4)^2))$	$\gamma = 0.0002$
Panel 4	Offsetting LSTAR Breaks at $T/5$ and $3T/4$	$d(t)=2+3/(1+\exp(\gamma(t-T/5))) -1.5/(1+\exp(\gamma(t-3T/4)))$	$\gamma = 0.05$
Panel 5	Reinforcing LSTAR Breaks at $T/5$ and $3T/4$	$d(t)=1.5/(1+\exp(\gamma(t-T/5))) +1.5/(1+\exp(\gamma(t-3T/4)))$	$\gamma = 0.05$
Panel 6	ESTAR Breaks at $T/5$ and $3T/4$	$d(t)=2+1.8(1-\exp(-\gamma(t-T/5)^2)) -1.5(1-\exp(-\gamma(t-3T/4)^2))$	$\gamma = 0.0003$

[Figure 1 Here]

The long-dashed line in each panel shows the results of estimating each series using the Bai-Perron (1998) procedure allowing for two breaks and the short-dashed line shows the results of the estimation with a Fourier approximation using two frequencies (*i.e.*, we set $n = 2$ so that $k = 1$ and $k = 2$) and a time trend. Hence, for the Fourier approximation, we regressed each y_t series on a constant, time trend and the four variables sine($2\pi t/T$), cosine($2\pi t/T$), sine($4\pi t/T$), and cosine($4\pi t/T$) where $T = 500$. Notice that the Bai-Perron procedure struggles to capture the forms of the smooth breaks in that it estimates each as a step-function.

While the Fourier approximation is designed to capture smooth breaks, Becker, Enders and Hurn (2006) and Enders and Lee (2011) show that the approximation can often capture the behavior of a series with sharp breaks. As shown in Panels 1 − 4 of Figure 2, a single frequency component (i.e., using only $k = 1$) can do reasonably well mimicking sharp discontinuous breaks although it struggles around the time of the breakpoint. A second frequency component can improve the fit. As shown in Panels 5 through 7, the Fourier approximation can do well even with sharp intercept and/or trend breaks if the series is continuous at the breakpoint. Panels 8 – 9 show the approximation with trend breaks that are discontinuous at the breakpoint.

[Figure 2 Here]

2.1 Power to Detect Structural Breaks

To be a bit more formal, Becker, Enders, and Hurn (2004) show that a Fourier approximation compares favorably to many types of structural break tests, even when the breaks are sharp. To use their Fourier test for, say $n = 3$, form the variables sine($2\pi t/T$), cosine($2\pi t/T$), sine($4\pi t/T$), cosine($4\pi t/T$), sine($6\pi t/T$), and cosine($6\pi t/T$), and regress y_t on them (and possibly a constant and a time trend). Gallant and Sousa (1991) show that if the frequencies are pre-specified, the joint distribution for the null hypothesis that all $a_k = b_k = 0$ is multivariate normal. If, instead, the single best frequency is estimated, the so-called Davies' (1987) problem arises since k is an unidentified nuisance parameter under the null hypothesis of no structural change. Let k^* denote the best-fitting single frequency component from the set $k = 1,\dots, 5$ (i.e., k^*

is the supremum value of k). Davies (1987) calculates the asymptotic distribution for the null hypothesis $a_{k^*} = b_{k^*} = 0$ and shows that the critical values depend only on the range of frequencies used in the test. Ludlow and Enders (2000) tabulate the critical values and we report (a variant) of their critical values for the null hypothesis $a_{k^*} = b_{k^*} = 0$ in Table 1 for $T = 100$, 200, and 500. For example, with a sample size of $T = 100$, the critical value of F for the null hypothesis $a_{k^*} = b_{k^*} = 0$ is 4.87 using a 95% confidence interval.

[Table 1 here]

In actuality, Becker, Enders, and Hurn (2004) used fractional frequencies (*i.e.*, non-integer values of k) and bootstrapped critical values to compare the Fourier approximation to the Andrews, Lee, and Ploberger (ALP) (1996) test, the CUSUM and $CUSUM^2$ test, and the UDmax and WDmax versions of the Bai and Perron test (1998). Table 2 shows selected results using the following six breaks:[2]

SB1: $y_t = x_t + \varepsilon_t$ $\quad t \le 40$
$y_t = 1.5x_t + \varepsilon_t$ $\quad t > 40$

SB2: $y_t = x_t + \varepsilon_t$ $\quad t \le 50$
$y_t = 1.5x_t + \varepsilon_t$ $\quad t > 50$

SB3: $y_t = x_t + \varepsilon_t$ $\quad t \le 20, t > 40$
$y_t = 1.5x_t + \varepsilon_t$ $\quad 20 < t \le 40$

SB4: $y_t = x_t + \varepsilon_t$ $\quad t \le 40, t > 55$
$y_t = 1.5x_t + \varepsilon_t$ $\quad 40 < t \le 55$

SB5: $y_t = x_t + \varepsilon_t$ $\quad t \le 20$
$y_t = 1.5x_t + \varepsilon_t$ $\quad 20 < t \le 40$
$y_t = 0.5x_t + \varepsilon_t$ $\quad t > 40$

SB6: $y_t = x_t + \varepsilon_t$ $\quad t \le 40$

[2] Note that these breaks were initially examined in originally analyzed in Clements and Hendry (1999).

$$y_t = 1.5x_t + \varepsilon_t \qquad 40 < t \le 55$$
$$y_t = 0.5x_t + \varepsilon_t \qquad t > 55$$

[Table 2 Here]

The values of x_t were drawn from a normal distribution with mean and variance equal to unity and a sample size $T = 60$. The ALP test examines all possible breakpoints occurring within the middle 90% of the data, and the Fourier approximation includes frequencies up to a maximum of six. The clear winners in this example are the ALP test and the Fourier approximation. The power for all of the tests deteriorates as the breakpoint moves from observation 40 to 50 and, although not shown in the table, the CUSUM and CUSUM^2 tests displayed comparatively low power for all breaks. When there is only one break present in the data generating process, SB1 and SB2, the ALP test performs slightly better than the Fourier test and much better than the BP tests. However, the Fourier test performs better than the ALP test for those processes that have two breakpoints occurring in the middle of the sample, SB3 to SB5, and the ALP test performs better if the breaks are late in the sample and asymmetric, SB6.

2.2 A Lagrange Multiplier Test

Suppose $d(t)$ is an intercept term of an p–th order autoregressive process $y_t = d(t) + a_1 y_{t-1} + \ldots + a_p y_{t-p} + \varepsilon_t$. To determine whether $d(t)$ is constant or is subject to a break, first estimate the series assuming the absence of a break. Then regress the residuals on a constant, y_{t-1} through y_{t-p}, and the low-valued trigonometric frequencies. If if you can reject the null hypothesis that the coefficients of the Fourier terms are jointly equal to zero, conclude there is a break. If you pre–select the frequencies, perform the test using a standard F-statistic. If you estimate a single frequency component, k^*, use the critical values reported in Table 1.

3. The Flexible Fourier Form, Unit Roots and Structural Breaks

Although it can often be difficult to differentiate between a unit root process and a process with a structural break, Perron (1989) shows how to modify the usual Dickey-Fuller test to account for a single sharp break at a known date and Perron (1997) shows how to account for the possibility that the break is endogenous. Lee and Strazicich (2005) show how to account for the possibility of two sharp endogeneous breaks. Unfortunately, it is difficult to tabulate critical values for more than two breaks and any such tests rapidly lose power as the number of breaks is allowed to expand. In contrast, Leybourne, Newbold and Vougas (1998) and Kapetanios, Shin and Snell (2003) develop unit-root tests allowing for a smooth break in the intercept of the process. To use either of these tests, it is necessary to assume a single smooth break with a known break date and functional form. The fact that the low frequency components of a Fourier approximation can capture the behavior of a series with multiple breaks suggests that it can be used in testing for a unit root in the presence of structural breaks. As suggested above, the methodology does not require the econometrician to specify the break dates, the number of breaks, or the form of breaks.

3.1 A Dickey-Fuller Type Test with a Fourier Approximation

Enders and Lee (2012) modify the Dickey-Fuller test to incorporate multiple low frequency components so as to mimic structural change. Consider the following data generating process (DGP):

$$\Delta y_t = d(t) + \rho y_{t-1} + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \varepsilon_t \tag{2}$$

where the deterministic term is a time dependent function denoted by $d(t)$, and ε_t is a stationary disturbance with variance σ_ε^2. The null hypothesis of a unit root (i.e., $\rho = 0$) can be tested directly if the functional form of $d(t)$ is known. However, in practice, $d(t)$ is usually unknown, and any test for $\rho = 0$ is problematic if the functional form of $d(t)$

is misspecified. This is where the Fourier approximation proves to be especially useful since $d(t)$ can be modeled as:

$$d(t) = \alpha_0 + \gamma t + \sum_{k=1}^{n} a_k \sin(2\pi kt/T) + \sum_{k=1}^{n} b_k \cos(2\pi kt/T) \tag{3}$$

where n represents the number of cumulative frequencies, k represents a particular frequency, and T is the number of usable observations.

The critical value for the null hypothesis $\rho = 0$ depends on the form of the deterministic regressors present in (3). Specifically, Enders and Lee (2011) show that the critical values depend on whether t is included as a regressor and on the frequencies used in the estimation.

The choice of lag length can be done using the standard general-to-specific method. The choice of the appropriate frequency components to include in the approximation can be done in several different ways. As a practical matter Enders and Lee (2012) recommend using only a single frequency (i.e., $k = 1$, 2, or 3) or a small number of cumulative frequencies (*i.e.*, $n = 1$, 2, or 3). The presence of many frequency components entails a loss of power and can lead to an over-fitting problem. Furthermore, n should be small to allow for the structural change to evolve gradually: it makes little sense to claim that a series reverts to a rapidly evolving mean.

If k is known, the appropriate critical values for the null hypothesis $\rho = 0$ are reported in the top portions Table 3 (if a time trend in included) and Table 4 (if no time trend in included). Although k is not likely to be known, Enders and Lee (2011) show that it can be consistently estimated. Successively using $k = 1$, 2, 3, ... the frequency component yielding the best fit yields a consistent estimate of k.

A simple pre-test for nonlinearity trend is to evaluate the F-statistic for the null hypothesis that the coefficients of the best-fitting frequency components are jointly equal to zero. Let k^* denote the frequency component yielding the best fit and let a_{k^*} and b_{k^*} denote the estimated coefficients of $\sin(2\pi k^* t/T)$ and $\cos(2\pi k^* t/T)$. The middle portions of Tables 3 and 4 report the critical values for the F-statistic for the null hypothesis $a_{k*} = b_{k*} = 0$. Note that rejecting this null hypothesis implies that the process is linear; in such circumstances, Enders and Lee (2012) recommend using a simple Dickey-Fuller test. If the null hypothesis is not rejected, it is possible to perform the test using k* or to use cumulative frequencies. If cumulative frequencies are used, the value of n can simply be pre-specified or be selected using a standard selection criterion such as the AIC or SBC. The lower portions of Tables 3 and 4 show the critical values of $\rho = 0$ when using cumulative frequencies.

[Tables 3 and 4 here]

3.2 Crude Oil Price Example (DF Test)

We applied the standard Dickey-Fuller test and the Fourier test to the monthly values of the real price of crude oil. The data was obtained from Enders and Holt (2012), and runs from 1960:1 through 2010:12. We followed their methodology and constructed the variable $loil_t$ as the logarithm of the price of oil (oil_t) deflated by the producer price index (PPI_t) normalized to the base year of 1996. Specifically:

$$loil_t = \log(100 * (\frac{oil_t / oil_{1996}}{PPI_t / PPI_{1996}})) .$$

We first estimated the series using the traditional Dickey-Fuller test:

$$\begin{aligned} \Delta loil_t = \underset{(2.35)}{0.0538} + \underset{(1.83)}{0.00005t} - \underset{(-2.37)}{0.0145 loil_{t-1}} + \underset{(3.84)}{0.1552 \Delta loil_{t-1}} - \underset{(-0.54)}{0.0219 \Delta loil_{t-2}} \\ + \underset{(2.74)}{0.1109 \Delta loil_{t-3}} \end{aligned} \tag{4}$$

Beginning with a maximum lag length of 12, we used the general-to-specific methodology to obtain the lag length of three. The t-statistic on the coefficient for $loil_{t-1}$ is -2.37 whereas the 5 percent and 10 percent critical values are -3.42 and -3.13, respectively. As such, we cannot reject the null hypothesis of a unit root in the real price of oil.

Although it might seem reasonable to use a unit root test allowing for endogenous breaks (such as Lee and Strazicich (2005)) it is doubtful that the such tests would be successful. Figure 3 (and Panel 4 of Figure 4) shows the time-series plot of $loil_t$ and the break dates found by the Bai-Perron (1996) method. To make a point, we show the series beginning with 1996; Panel 4 of Figure 4 shows the entire series. Note that the procedure does a poor job of selecting the break dates as the breaks are smooth. As such it seems reasonable to estimate the series with the Flexible Fourier Form so as to account for smooth breaks. Consider:

$$\begin{aligned}\Delta loil_t = {} & \underset{(6.17)}{0.357} + \underset{(4.68)}{0.0003t} - \underset{(-6.28)}{0.0996 loil_{t-1}} + \underset{(4.56)}{0.183\Delta loil_{t-1}} + \underset{(0.31)}{0.013\Delta loil_{t-2}} + \underset{(3.68)}{0.149\Delta loil_{t-3}} \\ & + \underset{(2.14)}{0.023\sin(2\pi t/T)} - \underset{(-4.61)}{0.034\cos(2\pi t/T)} - \underset{(-5.07)}{0.045\sin(4\pi t/T)} + \underset{(4.04)}{0.030\cos(4\pi t/T)} \\ & + \underset{(1.15)}{0.007\sin(6\pi t/T)} + \underset{(3.82)}{0.021\cos(6\pi t/T)}\end{aligned} \quad (13)$$

[Figure 3 Here]

The number of frequencies, $n = 3$, for the Fourier approximation was selected using the *AIC*. The important point is that the *t*-statistic on the coefficient for $loil_{t-1}$ is -6.28. According to Table 3, the 5 percent and 10 percent critical values are -5.57 and -5.29, respectively. Therefore, we reject the null hypothesis of a unit root and conclude that the real price of oil is stationary around a slowly evolving trend.

3.3 An LM Version of the Test

It is well-known that unit root tests relying on the Dickey-Fuller framework have extremely low power. The underlying reason for the low power is that the coefficients of the deterministic terms are poorly estimated. In order to produce a test with enhanced power, Enders and Lee (2011) develop a testing procedure based on the Lagrange Multiplier (LM) methodology. The idea is to estimate the coefficients of the deterministic terms using first-differences and then to detrend the series using these coefficients. As in a typical LM unit root test, the following regression is estimated in first differences:

$$\Delta y_t = \delta_0 + a_k \Delta \sin(2\pi kt/T) + b_k \Delta \cos(2\pi kt/T) + u_t \qquad (6)$$

Next, form $\tilde{S}_t$ as the detrended series is using the estimated coefficients from (6):

$$\tilde{S}_t = y_t - \psi - \tilde{\delta}_0 t - \tilde{a}_k \sin(2\pi kt/T) - \tilde{b}_k \cos(2\pi kt/T) \qquad (7)$$

where $\psi = y_1 - \tilde{\delta}_0 - \tilde{a}_k \sin(2\pi k/T) - \tilde{b}_k \cos(2\pi k/T)$.

Therefore, the testing regression based on the detrended series is

$$\Delta y_t = d_0 + \rho \tilde{S}_{t-1} + d_1 \Delta \sin(2\pi kt/T) + d_2 \Delta \cos(2\pi kt/T) + \varepsilon_t \qquad (8)$$

Note that equation (8) can be augmented with lagged valued of Δy_{t-i} in the presence of serial correlation. The coefficient of interest in (8) is ρ; if y_t is stationary, it must be the case that $\rho < 0$. The critical values for the null hypothesis $\rho = 0$ were tabulated by Enders and Lee (2011) and reported in Table 5. The critical values for a single pre-specified value of k (or a consistently estimated value of k) are reported in the top portion of Table 5. The center portion of the table shows the critical values of the F-test for the null hypothesis $a_{k^*} = b_{k^*} = 0$. The critical values for $\rho = 0$ for the LM test using cumulative frequencies are given in the lower portion of the table. For example, for $T = 100$ and $n = 2$, the 5% critical value for the null hypothesis $\rho = 0$ is -4.90.

[Table 5 Here]

3.4 Crude Oil Price Example (LM Test)

Enders and Holt (2012) applied the LM test with Fourier terms to the real price of oil. They used the AIC to select the number of cumulative frequencies and report that $n = 3$. Consider:

$$\Delta loil_t = d_0 - \underset{(-6.27)}{0.1}\tilde{S}_{t-1} + \sum_{k=1}^{3} [a_k \Delta \sin(2\pi kt/T) + b_k \Delta \cos(2\pi kt/T)] + \text{lagged changes} \qquad (9)$$

The t-statistic on the coefficient for $\tilde{S}_{t-1}$ using the Fourier approximation is -6.27. According to Table 5, the 5 percent and 10 percent critical values are -5.42 and -5.16, respectively. Therefore, they reject the null hypothesis of a unit root and conclude that the series is stationary around a slowly evolving mean.

3.5 A Test with the Null of Stationarity

The Dickey-Fuller and LM variants of the Fourier unit root test have the null hypothesis H_0: $\rho = 0$ and the alternative hypothesis H_a: $\rho < 0$. However, if the presumption is that a series is stationary, it seems natural to operate with the null hypothesis H_0: $\rho < 0$ and the alternative hypothesis H_a: $\rho = 0$. Moreover, since standard unit root tests suffer from low power, it often makes sense to confirm such tests with a procedure using the null of stationarity. Becker, Enders, and Lee (2006) (BEL) develop a modified version of the test by Kwiatkowski, Phillips, Schmidt, and Shin (1992) (KPSS) that tests the null of stationarity against the alternative of a unit root. As in the usual KPSS test, BEL compare the short-term error variance with the long-run error variance. Given that it is necessary account for the trigonometric terms, the BEL modified KPSS test statistic is:

$$\tau_{KPSS} = \frac{1}{T^2}\frac{\sum_{t=1}^{T}\tilde{S}_t(n)^2}{\tilde{\sigma}^2} \tag{10}$$

where $\tilde{S}_t(n) = \sum_{j=1}^{t}\tilde{e}_j$ and $\tilde{e}_j$ are the OLS residuals from the following regression:

$$y_t = \alpha + \beta t + \sum_{k=1}^{n}\{a_k \sin(2\pi kt/T) + b_k \cos(2\pi kt/T)\} + e_t \tag{11}$$

The long run variance $\tilde{\sigma}^2$ can be obtained by choosing a truncation parameter l and a set of weights w_j, $j = 1,\ldots, l$:

$$\tilde{\sigma}^2 = \tilde{\gamma}_0 + 2\sum w_j \tilde{\gamma}_j \tag{12}$$

where $\tilde{\gamma}_j$ is the j–th sample autocovariance of the residuals $\tilde{e}_t$. The critical values for this test depend on whether there is a deterministic trend in the estimating equation. Critical values from Becker, Enders, and Lee (2006) of τ_{KPSS} for a single frequency, k, are reported in the top portion of Table 6. The center portion of the tables contains the appropriate F-statistics for the null hypothesis $a_{k^*} = b_{k^*} = 0$. The critical values for cumulative frequencies are given in the lower portion of the table. For example, for $T = 100$, $n = 2$, and a trend term, the 5% critical value in Table 6 is 0.0318. The critical values of the right-hand-side of the table are applicable when a time trend is included in the estimating equation. Becker, Enders, and Lee (2006) discuss the properties of the test statistics and present simulated critical values. They also illustrate that these test have good power to detect U-shaped breaks and smooth breaks even near the end of the sample.

[Table 6 Here]

When Enders and Holt (2012) applied the stationarity test to the $loil_t$ series, they again found that the most appropriate number of frequencies, n, was three and that the τ_{KPSS} test statistic found from (10) was 0.024. According to Table 6, the 1% and 5% critical values for $T = 500$, $n = 3$, and a trend term, are 0.0265 and 0.0216, respectively. Therefore, they rejected the null of stationarity at the 5% significance level but not at the 1% significance level.

3.6 Size and Power of the Fourier Test Versus the Perron (1997) Test

Enders and Lee (2011) perform several Monte Carlo experiments to ascertain the finite sample properties of the LM version of the test. Although reported only in an earlier version of their paper, Enders and Lee (2011) show the consequences of using the Perron (1997) test when the breaks are actually smooth. First consider the following DGP:

$$y_t = \alpha_0 + \gamma \cdot t + a_k \sin(2\pi kt/T) + b_k \cos(2\pi kt/T) + e_t; k \le T/2 \tag{13}$$

$$e_t = \rho e_{t-1} + \varepsilon_t \tag{14}$$

The results shown in Table 7 are produced from generating 5,000 series using (13) and (14) with $k = 1$ and 2 and for various values of T, a_k, b_k, and ρ. As shown in row 1 of the table, for $k = 1$, $T = 500$, $a_1 = 0$, and $b_1 = 5$, the size of the Perron's (1997) endogenous break test with dummy variables is 4.3% and the power is only 24.9%. With the same parameter values, the Fourier test shows power in excess of 99% regardless of whether $n = 1$ or $n = 2$ is used. Perron's (1997) test continues to perform poorly when the DGP contains two frequencies (that is, when $k = 2$.) For example, if $k = 2$, $T = 500$, $a_2 = 0$, and $b_2 = 5$, the size of the test is 1.6% and the power is almost zero. On the other hand, the Fourier test with the same parameters and $k = 2$, gives a size of 4.8% and power of 95.2%. As indicated in the center portion of the table, it is not surprising that the Fourier test performs badly when the true DGP contains $k = 2$ but only $k = 1$ is used in the estimating equation. The point is that the omission of a frequency component present in the DGP reduces the power of any unit root test. Since the trigonometric components are orthogonal to each other, the use of $k = 1$ in the estimating equation does not capture the nature of a series with a value of $k = 2$ in the DGP.

[Table 7 Here]

It seems obvious that the Fourier test should work well in the presence of trigonometric components. Table 8 shows how the two tests perform in the presence of the following sharp breaks:

Break Type	**Function**	**$I_t = 1$ only if**
1. Intercept Break	$y_t = I_t d_1 + (1 - I_t) d_2 + \varepsilon_t$	$t \le T/2$
2. U-shaped Intercept	$y_t = I_t d_1 + (1 - I_t) d_2 + \varepsilon_t$	$t \le T/4$ or $t > 3T/4$
3. Intercept and Slope	$y_t = I_t (d_1 + t/T) + (1 - I_t)(d_2 + d_3 t/T) + \varepsilon_t$	$t \le T/2$
4. Temporary Change in Intercept and Slope	$y_t = I_t (d_1 + t/T) + (1 - I_t)(d_2 + d_3 t/T) + \varepsilon_t$	$t \le T/4$ or $t > 3T/4$

For each of the four breaks, Enders and Lee (2011) simulate 5,000 series using various values of d_i, $T = 200$ and 500, and $\rho = 1.0$ and 0.9 (the table reports only the results for $T = 500$). The results reveal that the Fourier test often performs better than the dummy endogenous break test. For example, when break type (4) is used with $d_1 = 6$, $d_2 = 0$, and $n = 1$, the Fourier test has size of 5.2% and power of 97%. Meanwhile, the dummy endogenous test using the same values has size of 6.7% and power of 82.1%. For (5) with $d_3 = 0.4$, and $n = 1$, the Fourier test has size of 5.1% and power of 99.6%. On the other hand, the dummy endogenous test using the same values has size of 7.5% and power of only 51.1%. The results show that the Fourier test provides better size for all of the simulated breaks, while often producing better power than the dummy endogenous test. This is a somewhat surprising result considering the dummy endogenous test is designed to model the types of sharp breaks used in the simulations. It is important to note that the Fourier test with $n = 1$ is superior to the test with $n = 2$. As suggested by Figure 2, a single frequency component with $k = 1$ is typically sufficient to capture the nature of structural breaks. Not surprisingly, although not reported here, the Fourier test generally performs better than Perron (1997) in the presence of the ESTAR and LSTAR breaks shown in Figure 1.

[Table 8 Here]

4. Modeling Commodity Prices

In an attempt to understand the recent run-up in commodity prices, Enders and Holt (2012) used a Fourier approximation to model the time series properties of the real prices of the following eighteen different commodities: maize, soy, wheat, sorghum, palm oil, rice, cotton, coffee, cocoa, sugar, beef, logs, rubber, petroleum, coal, ocean freight rates, and food. As described in Enders and Holt (2012), the data are monthly, and in most instances span the period 1960–2010. If y_t denotes one of the real commodity prices, their basic model can be written in the form of (2) and (3):

$$\Delta y_t \ = \ d(t) + \rho y_{t-1} + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \varepsilon_t$$

where

$$d(t) = \alpha_0 + \gamma t + \sum_{k=1}^{n} \alpha_k \sin(2\pi kt/T) + \sum_{k=1}^{n} \beta_k \cos(2\pi kt/T)$$

Assuming that $\rho < 0$ the underlying unconditional mean at time t is

$$E(y_t) \ = \ -\tilde{d}(t)/\rho, \tag{15}$$

As such, the nature of the model is such that structural breaks in $d(t)$ shift the mean of the series in question. For our purposes, the interesting feature of Enders and Holt (2012) is that they modeled each series using the Bai-Perron method and the Fourier method. As a prerequisite to modeling the series, Enders and Holt (2012) first reported the results of the Fourier LM test and the Fourier KPSS test as discussed above.

4.1 Unit Root and Stationarity Test Results

Table 9 shows selected results of the LM unit root test and the Becker, Enders and Lee (2006) stationarity test for six of the commodity prices. The second column of the table shows the number of frequencies selected (by minimizing AIC) up to a maximum of $n = 3$. For that number of frequencies, n, the next column shows the sample value of the τ_{LM} test statistic for the null hypothesis $\rho = 0$. For all of the commodities shown, it is possible to reject the null at the 5% level.

[Table 9 Here]

Columns 3 and 4 of the table contain the sample test statistics for the KPSS−type tests with and without the trend term. Recall that for $n = 3$, the 5% critical value is 0.0216 in the presence of a trend and 0.0729 without the trend. For maize, wheat, soybeans and rice it is not possible to reject the null hypothesis of stationarity at the 5% level. For cotton and petroleum, the null of stationarity cannot be rejected at the 1% level.

For those commodities where at least one of the tests indicated stationarity, Enders and Holt (2012) went on to estimate the shifting mean using the Bai-Perron method and the method using Fourier terms. Specifically, for each series, they began with the linear model and used the AIC to select the lag length p. For the Bai-Perron (1998) method, they set the maximum number of breaks at 9 and used a trim factor of 10%. The $UDmax$ test always rejected the null of no breaks, and the actual number of breaks used in the final estimation was that selected by the BIC.

For each commodity, Figure 4 shows the time path of the estimated breaks superimposed over the actual price series. For oil, the *last* break occurs in December 2004 (2004:12). This break precedes those for most of the other prices including maize (2006:08), soy (2007:04), rice (2008:01), cotton (2008:11), and wheat (2006:1). This is reasonably strong evidence in support of the claim that the rise in the price of oil reflected itself in a general rise in most other commodity prices. Nevertheless, if breaks are smooth, the Bai–Perron procedure is misspecified and the estimated break dates may not be informative of the actual change points in the series.

With almost 600 usable observations, Enders and Holt (2012) set $\max(n) = 10$ and estimated each series in the form of (2) and (3). They found that the AIC selected a relatively large value of n number in that most were at the upper bound. Although they point out that the number of smooth breaks in the data need not equal the number of frequencies used in the estimating equation, it might have been preferable to use the BIC (or some other parsimonious model selection criteria) to select n. Enders and Holt (2012) also report that they did not attempt to pare down the models by eliminating insignificant intermediate frequencies (e.g., for Maize, the value of k yielding the lowest AIC was $k = 6$ so that *sine* and *cosine* terms using frequencies $k = 1$ through $k = 6$. The long-dashed line in Figure 4 shows the fitted values using the Fourier approximation.

[Figure 4 Here]

5. Detecting Parameter Instability Using a Fourier Approximation

Although parameter instability can result from neglected structural change of the type discussed above, parameter instability can result from a number of factors including seasonality, stochastic parameter variation, or other neglected nonlinearities. Becker, Enders and Hurn (2006) show that the FFF is able to detect various types of parameter instability. If the $\{y_t\}$ sequence has a constant mean and is serially uncorrelated, for any value of k, the following regression equation should have no explanatory power:

$$\varepsilon_t = a_k \sin(2\pi kt/T) + b_k \cos(2\pi kt/T) + v_t \qquad (16)$$

Unlike the case of unit root tests, or KPSS–type tests, the null in (16) is that the $\{y_t\}$ series contains no omitted trigonometric terms (i.e., neglected breaks or nonlinearities). As such, there is no presumption as to which value or values of k should be selected. Nevertheless, time-varying parameters are more likely to manifest themselves in the upper portions of the spectral density function than are structural breaks. It is possible to estimate (16) for every value of k in a prespecified interval (say $1 \leq k \leq T/2$) and let k^* denote the best-fitting single frequency component. As discussed above, estimating k^* and testing the null $a_{k^*} = b_{k^*} = 0$, results in a Davies (1987) problem in that k^* is unidentified under the null hypothesis.[3] Hence, it is necessary to use the type of critical values reported in Table 1 or to bootstrap the test.

Becker, Enders and Hurn (2006) compare the Fourier test to the CUSUM test, CUSUM^2, the Nyblom (1989) test, and the Watson and Engle (1985) test for stochastic parameter variation for the following models of stochastic parameter variation. SPV1 uses a stationary autoregressive process for the time varying parameter (Watson and

[3] The Ludlow and Enders (2000) tabulated critical values are slightly different from those reported in Table 1.

Engle, 1985), while SPV2 uses a martingale (Nyblom, 1989). The third process, SPV3, is a bilinear specification (Lee *et al.*, 1993):

SPV1: $y_t = \beta_t y_{t-1} + \varepsilon_t \qquad \beta_t = 0.3 + 0.5\beta_{t-1} + \nu_t \qquad \varepsilon_t \sim N(0,1) \qquad \nu_t \sim N(0, 0.25)$

SPV2: $y_t = \beta_t y_{t-1} + \varepsilon_t \qquad \beta_t = \beta_{t-1} + \nu_t \qquad \varepsilon_t \sim N(0,1) \qquad \nu_t \sim N(0, 0.25)$

SPV3: $y_t = \beta_t y_{t-1} + \varepsilon_t \qquad \beta_t = 0.7\varepsilon_{t-2} \qquad \varepsilon_t \sim N(0,1)$

As shown in Table 10, the CUSUM and Nyblom (1989) tests perform poorly for all three processes while the CUSUM^2 test performs somewhat better than these two. The Watson and Engle (1985) test has decent power for the first two data generating processes, but very low power for the third. The power of the Flexible Fourier Form is strong in all three cases and the highest for two of the three cases.

Although not reported here, the power of a Fourier approximation to detect seasonal parameters is excellent due to the trigonometric terms included in the testing regression, which is identical to the use of seasonal dummy variables. However, the Fourier approximation does not require the econometrician to specify a fixed seasonal pattern. For this reason, the Fourier approximation can detect seasonality in the behavior of coefficients even when the exact cyclicality of the pattern is unknown.

Conclusion

A variant of Gallant's (1981) Flexible Fourier Form can serve multiple purposes in applied econometric time series: a unit root/stationarity test, a methodology for modeling a series, as well as, the ability to detect the various forms of parameter instability. Becker, Enders and Lee (2006) and Enders and Lee (2011) develop a stationarity test and a unit-root test that can be used in the presence of an unknown number of smooth breaks in the deterministic components of a series. The essential feature of the tests relies on the fact that a Fourier approximation forms an orthogonal basis able to capture the behavior of any integrable function. Hence, instead of estimating the number of breaks, the break dates and the form of the breaks, the methodology estimates the appropriate number of frequencies to include in the approximation. Becker, Enders and Lee (2006) and Enders and Lee (2011) both show that a small number of low frequency components can capture the behavior of a wide variety of breaks. The most important feature of the tests is that they have good size and power properties. It is also possible to use a Fourier approximation to detect time varying coefficients across different frequency ranges.

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Table 1: Supremum Critical Values for the Presence of the Fourier Coefficients

	Sample Size = 100				**Sample Size = 250**				**Sample Size = 500**			
	$k \leq 5$	$T/10$	$T/4$	$T/2$	$k \leq 5$	$T/10$	$T/4$	$T/2$	$k \leq 5$	$T/10$	$T/4$	$T/2$
99%	6.64	7.615	8.40	9.24	6.46	8.04	8.91	9.95	6.26	8.74	9.88	10.64
95%	4.87	5.56	6.61	7.38	4.68	6.31	7.31	8.11	4.58	6.99	8.00	8.52
90%	4.00	4.77	5.81	6.59	3.92	5.56	6.54	7.33	3.88	6.24	7.23	7.84

Table 2: Power of CUSUM, CUSUM^2, ALP, BP and *Trig*-test against structural breaks.

	p-value	SB1	SB2	SB3	SB4	SB5	SB6
ALP	0.01	0.320	0.174	0.069	0.121	0.633	0.105
	0.05	0.544	0.331	0.196	0.276	0.814	0.247
	0.10	0.675	0.450	0.323	0.375	0.885	0.364
Fourier	0.01	0.286	0.140	0.198	0.121	0.684	0.065
	0.05	0.485	0.294	0.408	0.284	0.877	0.171
	0.10	0.615	0.407	0.527	0.383	0.918	0.270
Udmax	0.01	0.127	0.058	0.039	0.062	0.332	0.028
	0.05	0.276	0158	0.143	0.165	0.559	0.100
	0.10	0.381	0.241	0.221	0.245	0.675	0.187
WDmax	0.01	0.109	0.044	0.047	0.056	0.326	0.030
	0.05	0.246	0.156	0.151	0.157	0.551	0.108
	0.10	0.349	0.232	0.230	0.240	0.662	0.192

Table 3: Critical Values for the *DF*-Version of the Test with a Trend

	***T* = 100**			***T* = 200**			***T* = 500**		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
k = 1	-4.95	-4.35	-4.05	-4.87	-4.31	-4.02	-4.81	-4.29	-4.01
k = 2	-4.69	-4.05	-3.71	-4.62	-4.01	-3.69	-4.57	-3.99	-3.67
k = 3	-4.45	-3.78	-3.44	-4.38	-3.77	-3.43	-4.38	-3.76	-3.43
Critical Values of max *F*(*k)**									
	12.21	9.14	7.78	11.70	8.88	7.62	11.52	8.78	7.53
Critical Values with Cumulated Frequencies									
n = 1	-4.95	-4.35	-4.05	-4.87	-4.31	-4.02	-4.81	-4.29	-4.01
n = 2	-5.68	-5.08	-4.78	-5.58	-5.02	-4.73	-5.5	-4.96	-4.69
n = 3	-6.33	-5.73	-5.42	-6.19	-5.63	-5.34	-6.1	-5.57	-5.29

Table 4: Critical Values Without a Trend

	***T* = 100**			***T* = 200**			***T* = 500**		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
k = 1	-4.42	-3.81	-3.49	-4.37	-3.78	-3.47	-4.35	-3.76	-3.46
k = 2	-3.97	-3.27	-2.91	-3.93	-3.26	-2.92	-3.91	-3.26	-2.91
k = 3	-3.77	-3.07	-2.71	-3.74	-3.06	-2.72	-3.70	-3.06	-2.72
Critical Values of max *F*(*k)**									
	10.35	7.58	6.35	10.02	7.41	6.25	9.78	7.29	6.16
Critical Values with Cumulated Frequencies									
n = 1	-4.18	-3.81	-3.49	-4.37	-3.77	-3.47	-4.35	-3.76	-3.46
n = 2	-5.16	-4.52	-4.19	-5.08	-4.48	-4.17	-5.02	-4.45	-4.14
n = 3	-5.79	-5.15	-4.80	-5.68	-5.08	-4.76	-5.61	-5.03	-4.72

Table 5: Critical Values of the LM-Version of the Test

	***T* = 100**			***T* = 200**			***T* = 500**		
	1%	5%	10%	1%	5%	10%	1%	5%	10%
k = 1	-4.69	-4.10	-3.82	-4.61	-4.07	-3.79	-4.57	-4.05	-3.78
k = 2	-4.25	-3.57	-3.23	-4.18	-3.55	-3.23	-4.13	-3.54	-3.22
k = 3	-3.98	-3.31	-2.96	-3.94	-3.30	-2.98	-3.94	-3.31	-2.98
Critical Values of max *F*(*k)**									
	11.79	8.80	7.50	11.32	8.60	7.34	11.03	8.37	7/18
Critical Values with Cumulated Frequencies									
n = 1	-4.69	-4.10	-3.82	-4.61	-4.07	-3.78	-4.57	-4.05	-3.78
n = 2	-5.49	-4.90	-4.61	-5.37	-4.84	-4.57	-5.30	-4.79	-4.52
n = 3	-6.18	-5.59	-5.29	-6.04	-5.48	-5.21	-5.95	-5.42	-5.16

Table 6: Critical Values of the KPSS-Version of the Test

	***T* = 100**			***T* = 500**			***T* = 100**			***T* = 500**		
	Model Without a Time Trend						**Model With a Time Trend**					
	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
k = 1	0.2699	0.1720	0.1318	0.2709	0.1696	0.1294	0.0716	0.0546	0.0471	0.0720	0.0539	0.0463
k = 2	0.6671	0.4152	0.3150	0.6615	0.4075	0.3053	0.2022	0.1321	0.1034	0.1968	0.1278	0.0995
k = 3	0.7182	0.4480	0.3393	0.7046	0.4424	0.3309	0.2103	0.1423	0.1141	0.2091	0.1404	0.1123
Critical Values of max *F*(*k)**												
	7.73	4.93	4.13	6.28	4.61	3.96	6.873	4.972	4.162	6.315	4.669	3.928
Critical Values with Cumulated Frequencies												
n = 1	0.2700	0.1735	0.1323	0.2696	0.1688	0.1290	0.0718	0.0548	0.0472	0.0714	0.0538	0.0462
n = 2	0.1638	0.1048	0.0800	0.1614	0.1023	0.0778	0.0399	0.0318	0.0282	0.0397	0.0312	0.0276
n = 3	0.1203	0.0769	0.0589	0.1157	0.0729	0.0553	0.0268	0.0222	0.0201	0.0265	0.0216	0.0193

Table 7: A Comparison of the Perron (1997) and Fourier Tests with Trigonometric Shifts

			Dummy-Endogenous Break Tests				Fourier Approximation: $n = 1$				Fourier Approximation: $n = 2$			
			$T = 200$		$T = 500$		$T = 200$		$T = 500$		$T = 200$		$T = 500$	
k	α_k	β_k	$\beta = 1.0$	$\beta = 0.9$	$\beta = 1.0$	$\beta = 0.9$	$\beta = 1.0$	$\beta = 0.9$	$\beta = 1.0$	$\beta = 0.9$	$\beta = 1.0$	$\beta = 0.9$	$\beta = 1.0$	$\beta = 0.9$
1	0	5	0.015	0.038	0.043	0.279	0.054	0.395	0.050	0.997	0.049	0.386	0.050	0.997
	3	0	0.035	0.188	0.046	0.868	0.053	0.396	0.049	0.996	0.049	0.386	0.050	0.997
	3	5	0.013	0.019	0.036	0.165	0.053	0.396	0.050	0.996	0.049	0.386	0.050	0.997
2	0	5	0.003	0.004	0.016	0.006	0.000	0.000	0.011	0.002	0.058	0.238	0.048	0.952
	3	0	0.021	0.088	0.042	0.547	0.002	0.015	0.026	0.559	0.058	0.238	0.048	0.952
	3	5	0.001	0.001	0.014	0.001	0.000	0.000	0.006	0.000	0.058	0.238	0.048	0.952

Notes to table:

1. For $k = 1$, the entries for the case of $n = 1$ are identical to those in Table 3 and the entries for $n = 2$ are identical to those in Table 4.

2. In the Perron (1997) tests, no trend was used since there is no trend in the DGP. The values for $\beta = 0.9$ denote size-adjusted powers of the relevant tests.

Table 8: Effects of Level and Trend Shifts

Breaks	d_1	d_2	d_3	Perron (1997) β = 1.0	Perron (1997) β = 0.9	Fourier (n = 1) β = 1.0	Fourier (n = 1) β = 0.9	Fourier (n = 2) β = 1.0	Fourier (n = 2) β = 0.9
Level									
Type 1	3	0		0.051	0.993	0.048	0.937	0.057	0.917
	6	0		0.050	0.968	0.047	0.879	0.069	0.935
Type 2	3	0		0.048	0.993	0.048	0.903	0.055	0.908
	6	0		0.052	0.970	0.045	0.716	0.067	0.821
Trend									
Type 3	0	0	0.2	0.051	0.996	0.048	0.951	0.065	0.781
	0	0	0.4	0.051	0.996	0.048	0.950	0.075	0.511
Type 4	0	0	0.2	0.052	0.996	0.049	0.950	0.054	0.838
	0	0	0.4	0.052	0.996	0.049	0.949	0.047	0.577

Note: For Type 1 and 2 breaks in the DGP, the endogenous break tests with level shifts were used. For Type 3 and 4 breaks in the DGP, the endogenous break tests with trend shifts were used. The values for β = 0.9 denote size-adjusted powers of the relevant tests.

Table 9: The Enders and Holt Unit Root and Stationary Tests

Commodity	n	τ_{LM}	τ_{KPSS}	τ_{KPSS} (Trend)	Last Break
Maize	3	−5.454	0.0210		2004:08
Wheat	3	−5.652	0.0189		2005:03
Soybeans	3	−6.378		0.041	2005:09
Rice	3	−5.432		0.022	2001:11
Cotton	3	−6.175	0.0221		
Oil	3	−6.221	0.0234		2002:07

Table 10: Power of tests for stochastic parameter variation

SPV1	CUSUM	CUSUM^2	Watson-Engle	Nyblom	Fourier
0.05	0.123	0.653	0.914	0.128	0.838
0.10	0.207	0.723	0.954	0.215	0.892
SPV2	**CUSUM**	**CUSUM^2**	**Watson-Engle**	**Nyblom**	**Fourier**
0.05	0.039	0.895	0.758	0.466	0.989
0.10	0.047	0.897	0.760	0.494	0.997
SPV3	**CUSUM**	**CUSUM^2**	**Watson-Engle**	**Nyblom**	**Fourier**
0.05	0.066	0.643	0.136	0.228	0.776
0.10	0.102	0.741	0.252	0.301	0.843

Sample size = 100. Power computed in 1000 repetitions using 400 bootstrap replications in each for the Watson-Engle and *Fourier*-tests

Figure 1: ESTAR and LSTAR Breaks

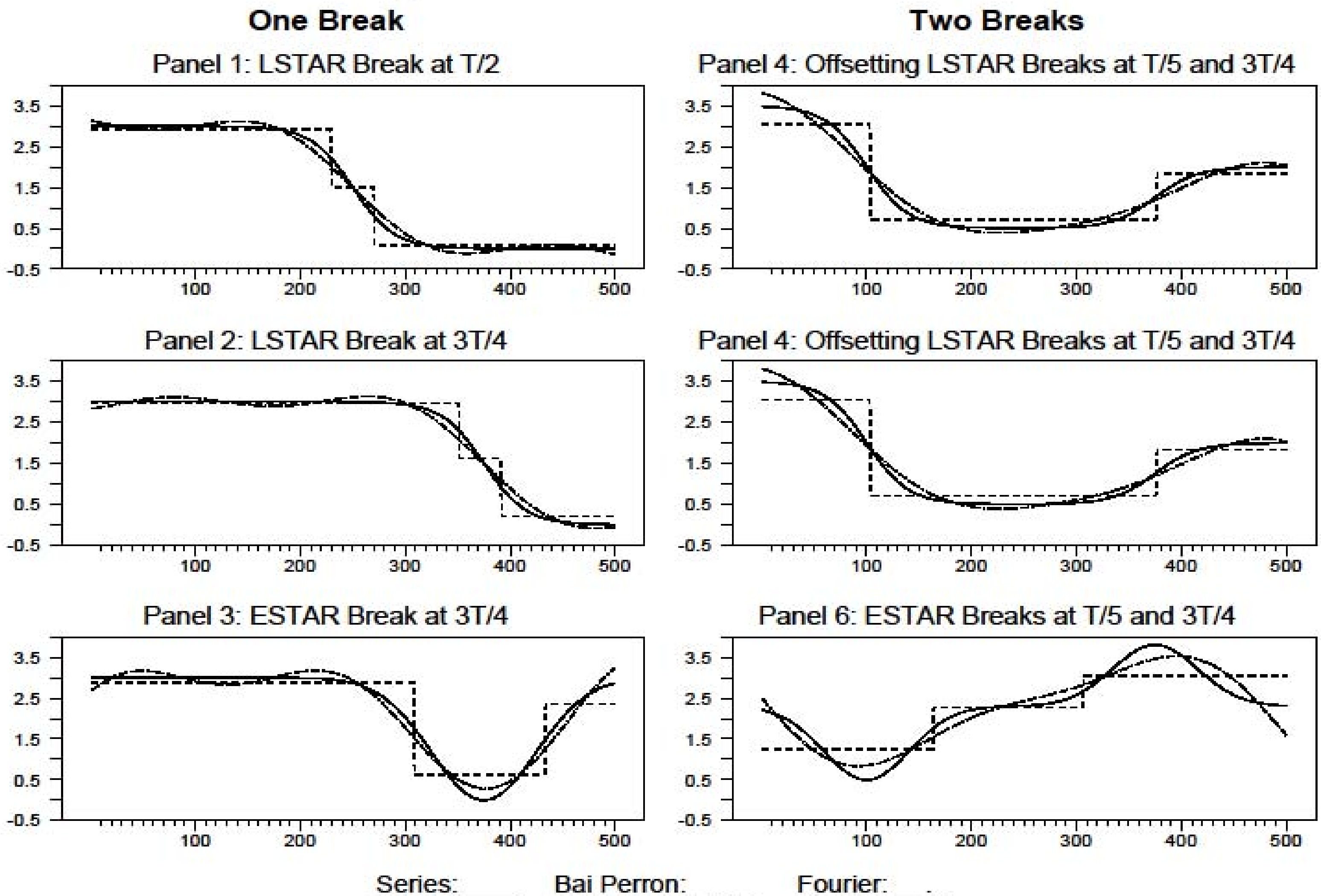
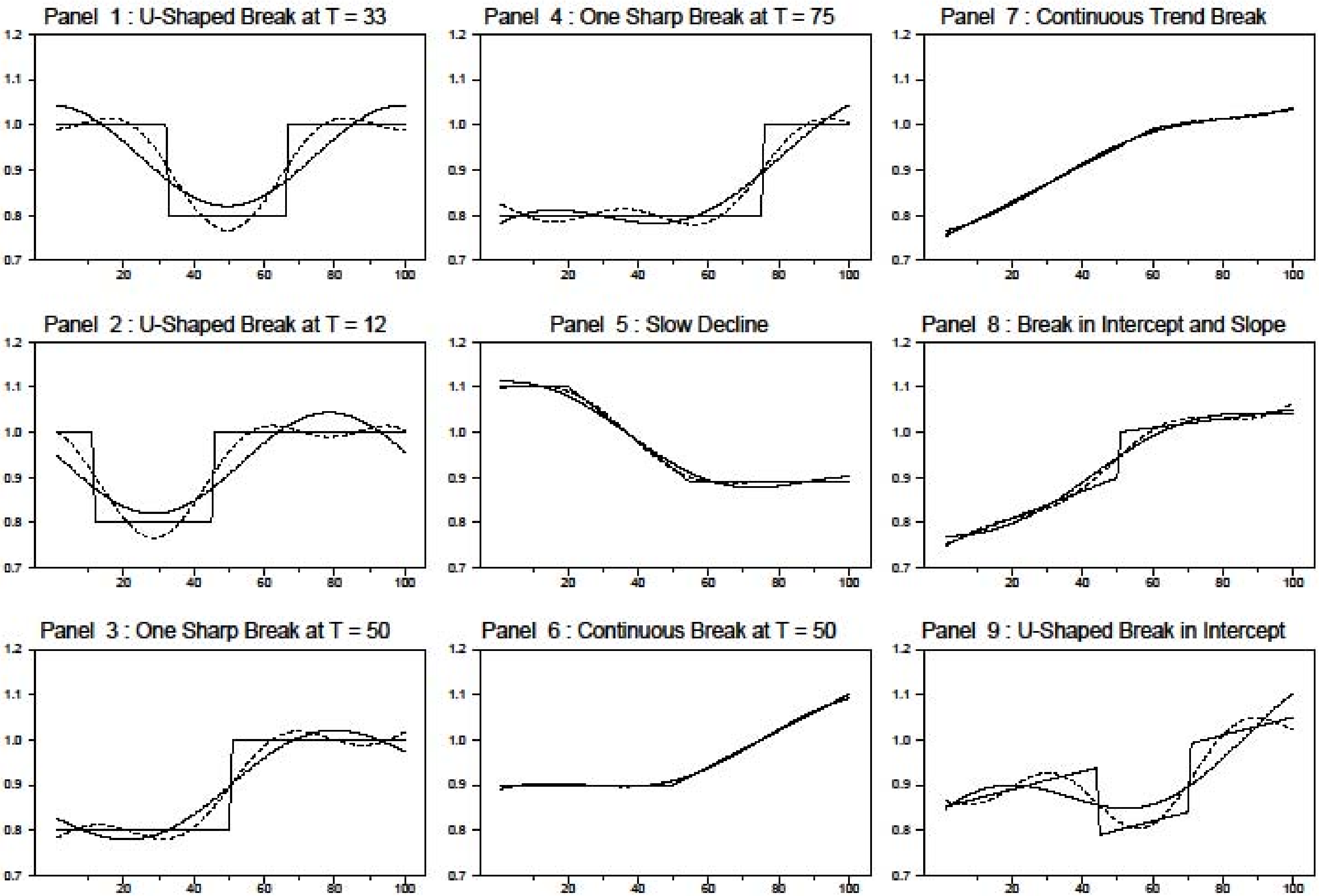


Figure 2: The Fourier Approximation and Sharp Breaks



Series: ____ 1 Frequency: _ _ _ 2 Frequencies: __._

Figure 3: The Bai-Perron Model and Oil

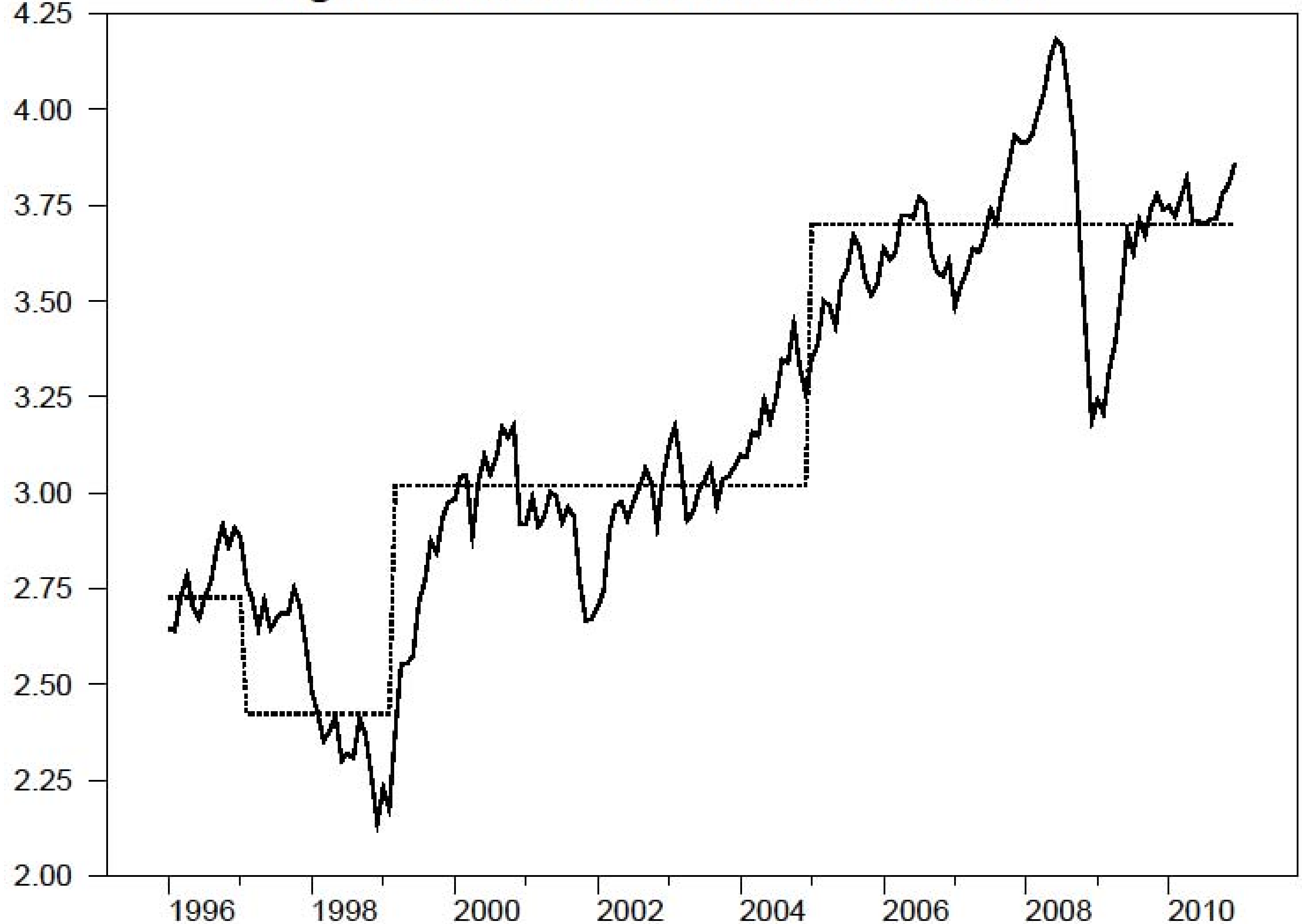
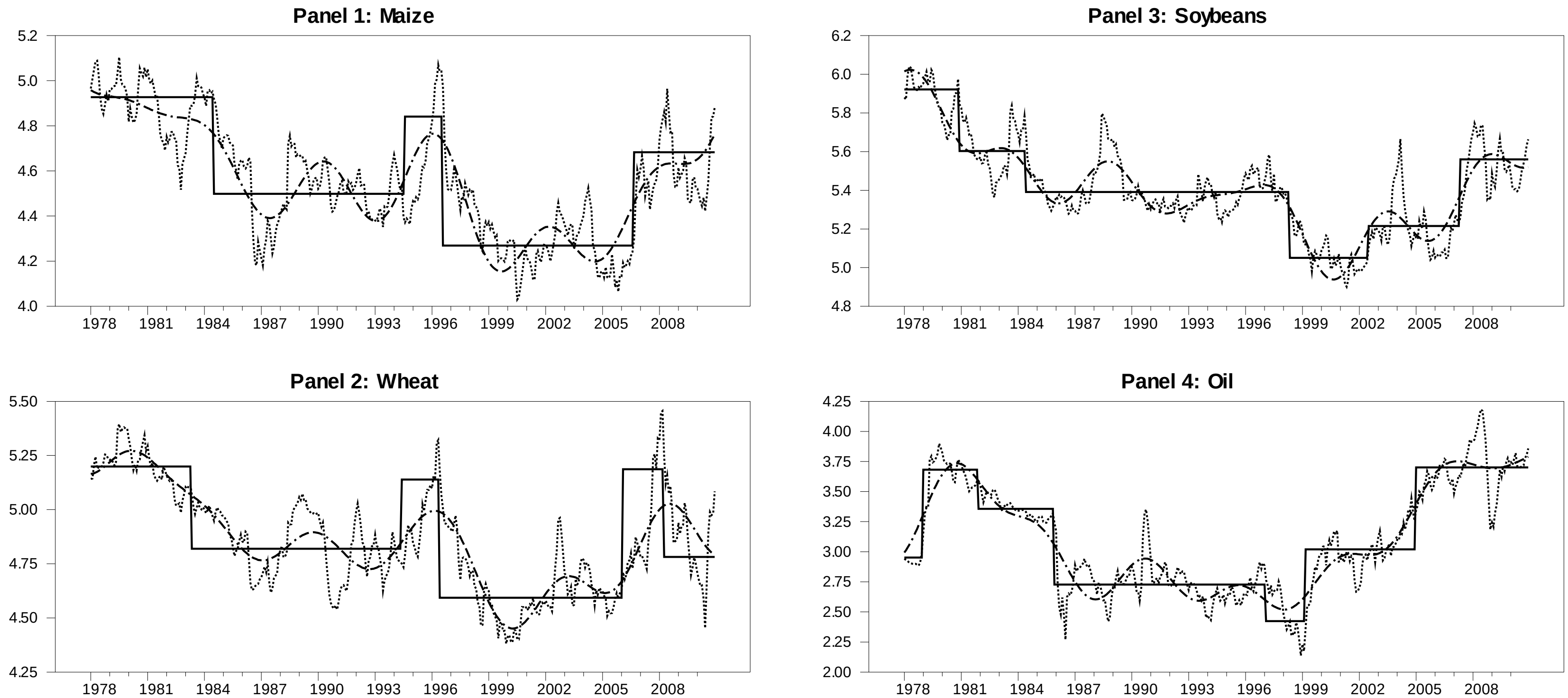


Figure 4: Modelling Commodity Prices



Actual .... Bai Perron ____ Fourier __ .