The Flexible Fourier Form and the Dickey-Fuller Type Unit Root Tests

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Abstract

We suggest a new unit-root test with a Fourier function in the deterministic term in a Dickey-Fuller type regression framework. Our suggested test can complement the Fourier LM and DF-GLS unit root tests. They have good size and power properties.

Keywords: Neglected nonlinearity, Fourier approximation.

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1. Introduction

Perron (1989) began a large and important literature by showing how to modify the standard Dickey-Fuller (DF) test to allow for a single structural break at a pre-specified change point. Since applied researchers can encounter series containing several potential breaks occurring at unknown dates, it is not surprising that much of the recent literature focuses on testing for a unit root in the presence of multiple endogenous breaks. Nevertheless, papers such as Lee and Strazicich (2003) do not consider more than two structural breaks since a test with many endogenous breaks is not likely to have much power. Moreover, Prodan (2008) shows that it can be quite difficult to properly estimate the number and magnitudes of multiple breaks, particularly when the breaks are of opposite sign. In the same vein, tests that would extend the nonlinear unit root tests of Leybourne, Newbold and Vougas(1998) and Kapetanios, Shin and Snell(2003) to allow for multiple smooth breaks do not seem especially promising.

To circumvent such problems, Enders and Lee (2012) and Rodrigues and Taylor (2012) propose a Fourier unit root test based on a variant of Gallant's (1981) Flexible Fourier Form (FFF). They follow Becker, Enders, and Hurn (2004) and Becker, Enders and Lee (2006) and illustrate that the essential characteristics of a series with one or more structural breaks can often be captured using a small number of low frequency components from a Fourier approximation. One important feature of the approximation method is that it is not necessary to assume that the number of breaks or the break dates are known *a priori.* Hence, instead of selecting specific break dates, the number of breaks, and the form of the breaks, the specification problem is transformed into incorporating the appropriate frequency components into the estimating equation.

In this paper, we extend the work of Enders and Lee (2012) and Rodrigues and Taylor (2012), and suggest a new Fourier unit root test in a Dickey-Fuller type regression framework. The

Fourier unit root test of Enders and Lee (2012) adopts the Lagrange Multiplier (LM) detrending method, while the Fourier test of Rodrigues and Taylor (2012) uses the so-called DF-GLS detending method. It is well known, however, that these detrending methods can result in a significant loss of power when the initial value is large. Not only is the standard DF methodology straightforward to use, DF-type unit root tests are free of this initial-value problem. Unlike Rodrigues and Taylor (2012), we also provide an *F*-test that can be used to pretest for the presence of nonlinearity. Such pretesting can be useful since utilizing the Fourier tests when no nonlinearity is present can result in a substantial loss of power.

2. Models with a Fourier function and Test statistics

We consider the following Dickey-Fuller test where the deterministic term is a time-dependent function denoted by $\alpha(t)$:

$$y_t = \alpha(t) + \rho\, y_{t-1} + \gamma \cdot t + \varepsilon_t \tag{1}$$

where ε_t is a stationary disturbance with variance σ_ε^2 and $\alpha(t)$ is a deterministic function of t. We are interested in testing the null hypothesis of a unit root (i.e., $\rho = 1$). When the form of $\alpha(t)$ is unknown, any test for $\rho = 1$ is problematic if $\alpha(t)$ is misspecified. As an approximation of the unknown functional form of $\alpha(t)$, consider the Fourier expansion

$$\alpha(t) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \sin(2\pi kt/T) + \sum_{k=1}^{n} \beta_k \cos(2\pi kt/T);\ n \le T/2 \tag{2}$$

where n represents the number of frequencies contained in the approximation, k represents a particular frequency, and T is the number of observations.

Obviously, if $\alpha_1 = \beta_1 = \ldots = 0$, the process is linear and the traditional unit root testing methodologies are appropriate. However, if there is a break or nonlinear trend, at least one Fourier frequency must be present in the data-generating process. As pointed out by Gallant (1981), most other approximations, such as a Taylor series, are valid at a particular point in the sample space. An important advantage of a Fourier approximation is that it is a global, rather than a local, approximation. As a practical matter, it is not possible to use a large value of n in a regression framework. The use of many frequency components uses degrees of freedom and can lead to an over-fitting problem. Thus, instead of positing the specific form of $\alpha(t)$, the issue is to select the proper frequencies to include in (2). For the time being, suppose we use only a single frequency k and consider the testing regression

$$\Delta y_t = \rho\, y_{t-1} + c_1 + c_2\, t + c_3 \sin(2\pi kt/T) + c_4 \cos(2\pi kt/T) + e_t \qquad (3)$$

We let τ_{DF_t} denote the t-statistic for the null hypothesis $\rho = 0$ in (3). The asymptotic properties of the DF version tests are not different from those of the LM version of the test, and we chose not to show the asymptotic distribution. The key point is that the critical values for the null hypothesis of a unit root will depend only on the frequency k and the sample size T as in the other version tests. However, they do not depend on the coefficients of the Fourier terms or other deterministic terms. Thus, we can tabulate critical values using simulations. Critical values of τ_{DF_t} are reported in Table 1a. If the researcher is willing to specify the value of k, the test can be conducted directly using these critical values. If the value of k is estimated, the test for a break can be performed as follows:

Step 1: Estimate (3) for all integer values of k such that $1 \le k \le 5$. The regression with the smallest SSR yields $\hat{k}$. If the residuals exhibit serial correlation, augment (3) with lagged values of Δy_t.

Step 2: Pretesting for nonlinearities can be done. For this, perform the usual F-test for the null hypothesis $c_3 = c_4 = 0$. The distribution of the F-statistic is non-standard when the unit-root null is imposed on the DGP. Thus, we can use the critical values shown in Table 1a at the bottom with the label "Critical Values of $F(\hat{k})$". For example, with a sample size of 100, the critical value at the 5% level is 9.14. If the sample value of F is less than the critical value, the null hypothesis of a linear trend is not rejected. At this circumstance, we recommend performing the usual linear Dickey-Fuller test.

In some circumstances there is no need to include a deterministic time trend in (3). Such models allow for an unknown form of level-shifts but preclude the presence of a linear trend. Then, it is possible to increase the power of the test by excluding an unnecessary time trend from the estimating equation. For these situations, we obtained the appropriate critical values by excluding the trend function t from the estimating equation. These tests excluding a linear trend are generally more powerful than the tests assuming the presence of a linear trend. We refer to this test as τ_{DF_C}. The critical values obtained from our Monte Carlo simulations are shown in Table 1b. For example, for $T = 100$, and $k = 1$, the 5% critical value for the null hypothesis $\rho = 0$ is –3.81. Table 1b shows the critical values for the F-test for the null hypothesis $c_3 = c_4 = 0$ for each value of k. When k is treated as an unknown, it is necessary to use the reported supremum $F(\hat{k})$ values.

We next consider the Fourier tests with multiple frequencies in the Fourier function as in Enders and Lee (2012). The motivation of using multiple frequencies is that they can provide a more precise approximation. If cumulated frequencies are used, we denote the resulting tests as $\tau_{DF\_t}(n)$ and $\tau_{DF\_C}(n)$, respectively. The appropriate critical values for $\tau_{DF\_t}(n)$ and $\tau_{DF\_C}(n)$ for $n = 1$, ..., 5 are contained in Tables 2a and 2b. Although using cumulative frequencies can offer more precise estimates, there is a caveat that they can yield significant loss of power due to over-fitting of the data.

3. The Monte Carlo experiments

We now examine the performance of the suggested Fourier test using simulation experiments. All Monte Carlo simulations are performed using 20,000 replications at the five percent significance level. First, we show that the Fourier test statistic τ_{DF_t} will depend on k but will be invariant to the magnitudes of the α_k and β_k. The results in Table 3 show that the size of our tests is close to 5% in all cases with different values of k, α_k and β_k, regardless of the sample sizes. When the sample size is small, the power of the test is low. But, increasing the sample size to $T = 200$ or 500 rapidly increases the power of the test towards 1.0. This result implies that the test is consistent. As expected, the power of the τ_{DF_C} test can be quite high as the trend is excluded from the testing regression. For example, as shown in Table 3, when $T = 100$, $k = 1$, $\alpha_1 = 0$, $\beta_1 = 5$, and $\rho = 0.9$, the power of τ_{DF_C} test 18.0% and exceeds 30% for $k = 2$ through 5.

We next examine the cases where k is estimated from the data. We adopt a two-step procedure where we first determine if a nonlinear trend exists or not and test for a unit root in the second step. We test the null hypothesis $c_3 = c_4 = 0$ in (4) by using the values of $F(\hat{k})$ reported in the lower portion of Table 1a or Table 1b. If the null hypothesis of a liner trend is not rejected, we apply usual linear Dickey-Fuller unit-root test. In Table 4, we report the simulation results for a sample size of 200. They show that the test has fairly good size and power properties. We observe that the power is the greatest when $\alpha_k = \beta_k = 0$, where the data-generating process (DGP) is actually linear. Also, note that these results are insensitive to the actual value of k used in the DGP. Unreported experiment (available from us on request) shows that increasing the sample size to T = 500 greatly improves the power of the test.

One attractive feature of the test is that it yields an approximation to the form of various break(s) present in the DGP. As such, it is important to document that the estimated frequency seems to be quite close to the actual frequency present in the DGP. The right-hand-portion of Table 4 reports the proportions of the frequencies estimated for a variety series. The test is most likely to select the correct frequency when α_k and β_k are large (since these increase the importance of the trigonometric components), when ρ is small (since a unit root is a "low frequency" event) and when T is large.

Next, we examine the test which includes cumulative frequencies in the estimating equation. The Monte Carlo results reported in Table 5 indicate that our $\tau_{DF}(n)$, and $\tau_{DF\_C}(n)$ tests are correctly sized when cumulative frequencies are used. The problem is that the power of the test diminishes rapidly as additional frequency components are added to the estimating equation. In essence, an over-fitting phenomenon occurs when a large number of frequency components are included in the estimating equation. For example, if the number of frequency components increases past $n = 3$, the power of the test deteriorates rapidly even for large T. Note that if there is no trend in the DGP, the power of the $\tau_{DF\_C}(n)$ test can far exceed the power of the other two tests. As such, we recommend that a small value of n, such as $n = 1$ or $n = 2$, be used.[1]

4. Concluding Remarks

In this paper, we suggest the Dickey-Fuller version of the Fourier unit root tests which can be useful in the presence of unknown multiple breaks in a nonlinear fashion. Since we use a parsimonious number of parameters, we can avoid the problem of losing power that can be found from unit root tests using too many dummy variables. Our suggested tests can complement the LM version of Enders and Lee (2012) or the DF-GLS version test of Rodrigues and Taylor (2012). In

[1] We also experimented with using the BIC to select the appropriate value of n. In a number of cases, estimating n using the BIC worked quite well. However, we were not surprised to find estimating n often leads to substantial size distortions since the low frequency components can "filter out" the unit root.

particular, the DF version tests are robust in the presence of a large initial value. Moreover, we also suggest the DF version tests with level shifts when a linear trend is absent. In such cases, the LM version or the DF-GLS version tests can be less powerful than those reported here.

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Moreover, results in the presence of serially correlated errors are not essentially the same as those from usual DF tests. We omit these results here but they are available upon request.

Table 1a: Critical Values for τ_{DF_t}

	T = 100			*T* = 200			*T* = 500			T = 2500		
k	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
Critical Values of τ_{DF_t}												
1	-4.95	-4.35	-4.05	-4.87	-4.31	-4.02	-4.81	-4.29	-4.01	-4.80	-4.27	-4.00
2	-4.69	-4.05	-3.71	-4.62	-4.01	-3.69	-4.57	-3.99	-3.67	-4.58	-3.98	-3.67
3	-4.45	-3.78	-3.44	-4.38	-3.77	-3.43	-4.38	-3.76	-3.43	-4.38	-3.75	-3.43
4	-4.29	-3.65	-3.29	-4.27	-3.63	-3.31	-4.25	-3.64	-3.31	-4.24	-3.63	-3.30
5	-4.20	-3.56	-3.22	-4.18	-3.56	-3.24	-4.18	-3.56	-3.25	-4.16	-3.55	-3.24
Critical Values of $F(\hat{k}) = Max\ F(k)$												
	7.78	9.14	12.21	7.62	8.88	11.70	7.53	8.76	11.52	7.50	8.71	11.35

Table 1b: Critical Values for τ_{DF_C} Without a Linear Trend

	T = 100			*T* = 200			*T* = 500			T = 2500		
k	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
Critical Values of τ_{DF_C}												
1	-4.42	-3.81	-3.49	-4.37	-3.78	-3.47	-4.35	-3.76	-3.46	-4.31	-3.75	-3.45
2	-3.97	-3.27	-2.91	-3.93	-3.26	-2.92	-3.91	-3.26	-2.91	-3.89	-3.25	-2.90
3	-3.77	-3.07	-2.71	-3.74	-3.06	-2.72	-3.70	-3.06	-2.72	-3.69	-3.05	-2.71
4	-3.64	-2.97	-2.64	-3.62	-2.98	-2.65	-3.62	-2.97	-2.66	-3.61	-2.96	-2.64
5	-3.58	-2.93	-2.60	-3.55	-2.94	-2.62	-3.56	-2.94	-2.62	-3.53	-2.93	-2.61
Critical Values of $F(\hat{k}) = Max\ F(k)$												
	6.35	7.58	10.35	6.25	7.41	10.02	6.16	7.29	9.78	6.11	7.25	9.72

Table 2a: Critical Values of τ_{DF_t} (n) Using Cumulated Frequencies

	T = 100			*T* = 200			*T* = 500			T = 2500		
n	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
1	-4.95	-4.35	-4.05	-4.87	-4.31	-4.02	-4.81	-4.29	-4.01	-4.80	-4.27	-4.00
2	-5.68	-5.08	-4.78	-5.58	-5.02	-4.73	-5.50	-4.96	-4.69	-5.48	-4.95	-4.68
3	-6.33	-5.73	-5.42	-6.19	-5.63	-5.34	-6.10	-5.57	-5.29	-6.04	-5.54	-5.27
4	-6.94	-6.31	-6.00	-6.73	-6.18	-5.89	-6.61	-6.10	-5.83	-6.57	-6.05	-5.79
5	-7.52	-6.86	-6.54	-7.24	-6.68	-6.39	-7.10	-6.58	-6.32	-7.03	-6.53	-6.26

Table 2b: Critical Values of τ_{DF_C} (n) Using Cumulated Frequencies

	T = 100			*T* = 200			*T* = 500			T = 2500		
n	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
1	-4.42	-3.81	-3.49	-4.37	-3.78	-3.47	-4.35	-3.76	-3.46	-4.31	-3.75	-3.45
2	-5.16	-4.52	-4.19	-5.08	-4.48	-4.17	-5.02	-4.45	-4.14	-4.98	-4.42	-4.12
3	-5.79	-5.15	-4.80	-5.68	-5.08	-4.76	-5.61	-5.03	-4.72	-5.58	-5.01	-4.69
4	-6.40	-5.71	-5.35	-6.23	-5.60	-5.28	-6.14	-5.55	-5.23	-6.09	-5.51	-5.19
5	-6.95	-6.23	-5.86	-6.73	-6.09	-5.75	-6.61	-6.02	-5.70	-6.54	-5.97	-5.66

Table 3: Finite Sample Performance of the Tests with Known Frequencies

DGP			T = 100		T = 200		T = 500	
k	α_k	β_k	$\rho = 1.0$	$\rho = 0.9$	$\rho = 1.0$	$\rho = 0.9$	$\rho = 1.0$	$\rho = 0.9$
The τ_{DF_t} Version of the Test								
1	0	5	0.048	0.104	0.050	0.321	0.051	0.994
	3	0	0.051	0.105	0.050	0.318	0.049	0.994
	3	5	0.051	0.108	0.050	0.318	0.050	0.993
2	0	5	0.049	0.158	0.049	0.479	0.054	0.999
	3	0	0.051	0.158	0.049	0.477	0.051	0.999
	3	5	0.049	0.153	0.052	0.478	0.050	0.999
3	0	5	0.048	0.181	0.048	0.565	0.049	1.000
	3	0	0.050	0.176	0.048	0.567	0.047	1.000
	3	5	0.050	0.182	0.047	0.566	0.050	1.000
The τ_{DF_C} Version of the Test								
1	0	5	0.051	0.180	0.049	0.525	0.050	1.000
	3	0	0.049	0.182	0.050	0.521	0.050	1.000
	3	5	0.050	0.181	0.049	0.526	0.052	1.000
2	0	5	0.051	0.312	0.050	0.796	0.048	1.000
	3	0	0.050	0.309	0.052	0.797	0.051	1.000
	3	5	0.051	0.314	0.052	0.797	0.049	1.000
3	0	5	0.048	0.327	0.049	0.846	0.050	1.000
	3	0	0.052	0.325	0.049	0.850	0.050	1.000
	3	5	0.049	0.326	0.053	0.847	0.051	1.000

Table 4: Finite Sample Performance of τ_{DF_t} Using the $F(\hat{k})$ Test

(a) Size with T = 200, ρ = 1

DGP k	α_k	β_k	5% Rej. rate	10% Rej. rate	Linear	1	2	3	4	5	6 ≥
					Relative frequency of the selected values of k						
1	0	0	0.080	0.138	0.94	0.04	0.01	0.00	0.00	0.00	0.00
	0	5	0.049	0.091	0.89	0.10	0.00	0.00	0.00	0.00	0.00
	3	0	0.068	0.121	0.92	0.07	0.01	0.00	0.00	0.00	0.00
	3	5	0.053	0.092	0.82	0.17	0.00	0.00	0.00	0.00	0.00
2	0	0	0.076	0.136	0.95	0.04	0.01	0.00	0.00	0.00	0.00
	0	5	0.045	0.087	0.64	0.01	0.35	0.00	0.00	0.00	0.00
	3	0	0.055	0.098	0.87	0.02	0.11	0.00	0.00	0.00	0.00
	3	5	0.045	0.084	0.46	0.00	0.54	0.00	0.00	0.00	0.00

(b) Power with T = 200, ρ = 0.9

DGP k	α_k	β_k	5% Rej. rate	10% Rej. rate	Linear	1	2	3	4	5	6 ≥
					Relative frequency of the selected values of k						
1	0	0	0.641	0.809	0.99	0.01	0.00	0.00	0.00	0.00	0.00
	0	5	0.232	0.256	0.74	0.26	0.00	0.00	0.00	0.00	0.00
	3	0	0.331	0.490	0.95	0.04	0.00	0.00	0.00	0.00	0.00
	3	5	0.292	0.384	0.58	0.42	0.00	0.00	0.00	0.00	0.00
2	0	0	0.644	0.812	0.99	0.01	0.00	0.00	0.00	0.00	0.00
	0	5	0.449	0.585	0.28	0.00	0.72	0.00	0.00	0.00	0.00
	3	0	0.259	0.358	0.78	0.00	0.22	0.00	0.00	0.00	0.00
	3	5	0.465	0.657	0.09	0.00	0.91	0.00	0.00	0.00	0.00

Table 5: Size and Power Using Cumulative Frequencies

	T = 100		T = 200		T = 500	
n	ρ = 1	ρ = 0.9	ρ = 1	ρ = 0.9	ρ = 1	ρ = 0.9
	Finite Sample Performance of $\tau_{DF}(n)$					
1	0.053	0.106	0.049	0.321	0.049	0.994
2	0.053	0.083	0.050	0.197	0.051	0.940
3	0.048	0.069	0.049	0.137	0.051	0.804
4	0.050	0.063	0.048	0.113	0.050	0.653
5	0.047	0.061	0.046	0.096	0.051	0.515
	Finite Sample Performance of $\tau_{DF\_C}(n)$					
1	0.050	0.179	0.050	0.541	0.050	1.000
2	0.053	0.156	0.050	0.365	0.049	0.990
3	0.051	0.139	0.049	0.277	0.050	0.951