Stationarity tests with unattended nonlinearity

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Abstract

The paper develops a test with the null of stationarity that allows for the possibility of an unknown number of structural breaks, or other nonlinearities, in the data-generating process. The test is based on the fact that the behavior of a breaking process can often be captured using a single frequency component of a Fourier approximation. Hence, instead of selecting specific break dates, the number of breaks, and the form of any nonlinearities, the specification problem is transformed into selecting a low frequency component to include in the estimating equation. Our proposed test does not exhibit any serious size distortions, and shows reasonable power. The appropriate use of the test is illustrated using real exchange rates in the post-Bretton Woods period.

Keywords: Structural breaks, unit roots, Fourier approximation.

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1. Introduction

The aim of this paper is to develop a test with the null hypothesis of stationarity that allows for an unknown number of structural breaks of unknown functional form(s).[1] Tests for stationarity have been suggested by many authors including Nyblom (1986), Kwiatkowski, Phillips, Schmidt and Shin (1992, KPSS hereafter), and Leybourne and McCabe (1994). It is well known that stationarity tests can be invalid when a researcher ignores structural breaks present in the data-generating process (DGP). In an attempt to rectify this problem, Busetti and Harvey (2001), Presno and Lopez (2003), Busetti and Taylor (2003), and Kurozumi (2002), modify the various KPSS-type tests by including dummy variables to capture changes in the level and trend.

The existing stationarity tests with a break assume, *a priori,* the presence of a single structural break in the level and/or the trend of the series in question. If it is impossible to rule out additional breaks, this single-break assumption is problematic. Unfortunately, the problem in devising a test with multiple breaks is that the distributions are difficult to compute because the appropriate test statistics depend on the location of breaks. In principle, it is possible to obtain critical values for all possible break combinations if the break dates are known. However, in applied work, if the number of breaks is unknown, the break dates are also likely to be unknown.

Another difficulty lies in the fact that a break occurring at time *t* need not manifest itself contemporaneously. Even major breaks, such as the stock market crash of 1929 and the oil price shocks of the 1970s, did not display their full impacts immediately. As such, there is a growing literature to test for a unit root in the presence of nonlinearities. For example, Luukkonen,

[1] We use the term 'unit-root test' to refer to any test with the null hypothesis of a unit root and an alternative of stationarity. Similarly, we use 'stationarity test' to describe a test with the null hypothesis of stationarity and an alternative of a unit root.

Saikkonen, and Teresvirta (1988), Kapetanios, Shin and Snell (2003), and Leybourne, Newbold and Vougas (1998) develop tests for a unit root allowing for a single break such that the deterministic component of the model is a smooth transition process. To our knowledge, such an approach has not yet been adopted in testing for stationarity.

Without any information concerning the actual nature of the break, positing sharp breaks or a specific type of smooth transition process can seem somewhat *ad hoc*. Using an incorrect specification for the form and/or number of breaks might be worse than ignoring the breaks altogether. However, Gallant (1984), Davies (1987), Gallant and Souza (1991), and Becker, Enders and Hurn (2004) show that a Fourier approximation can often capture the behavior of an unknown function even if the function itself is not periodic. In the next section, we illustrate this point by showing that a series with multiple structural breaks of unknown form can often be captured using a single frequency component of a Fourier approximation. Although the single frequency component can detect sharp breaks, it is designed to work best when breaks are gradual.[2] Moreover, our approximation method can detect u-shaped breaks and smooth breaks located near the end of a series. Hence, instead of selecting specific break dates, the number of breaks, and the form of the breaks, the specification problem is transformed into selecting the proper frequency component to include in the estimating equation.

Since we use a trigonometric term to capture unknown nonlinearities, our work parallels the unit root tests suggested by Bierens (1997) and Enders and Lee (2004). Here, we develop

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[2] In point of fact, it is difficult to distinguish between a structural break and certain types of nonlinearities. Clearly, a series with a break can be viewed as a special case of a process that is nonlinear in its parameters. As such, our approach can be viewed as an attempt to provide a general procedure to approximate unknown nonlinear components. The phrase 'unattended nonlinearity' in the title of the paper reflects the more general nature of our test. Nevertheless, the discussion in the text focuses on structural breaks.

KPSS-type stationarity test since tests with the null of a unit root have low power with stationary, but persistent, data. The problem is exacerbated when a theory, such as purchasing power parity or the convergence of growth rates across nations, can be more naturally tested under the null of stationarity. Moreover, stationarity tests are useful since they can be used to confirm results from unit root tests with a stationary alternative such as the Dickey-Fuller (DF) or Phillips-Perron tests. Throughout the paper, "$\rightarrow$" indicates weak convergence as T approaches ∞.

2. Test Statistics

We consider the following DGP:

$$y_t = X_t'\beta + Z_t'\gamma + r_t + \varepsilon_t \tag{1}$$

$$r_t = r_{t-1} + u_t$$

where ε_t are stationary errors and u_t are *iid* $(0, \sigma_u^2)$. Here, we use $X_t = [1]$ for a level stationary process for y_t and $X_t = [1, t]'$ for a trend stationary process. We choose $Z_t = [\sin(2\pi kt/T), \cos(2\pi kt/T)]'$ to capture a break (or other type of unattended nonlinearity) in the deterministic term, where k is the frequency and T is the sample size. Under the null hypothesis $\sigma_u^2 = 0$, so that the process described by (1) is stationary.

The rationale for selecting $Z_t = [\sin(2\pi kt/T), \cos(2\pi kt/T)]'$ is based on the fact that a Fourier expansion is capable of approximating absolutely integrable functions to any desired degree of accuracy. Let $\alpha(t)$ denote a function with an unknown number of breaks of unknown form(s). Regardless of the nature of the breaks, under very weak conditions, $\alpha(t)$ can be exactly represented by a sufficiently long Fourier series. To keep the problem tractable, we consider a Fourier approximation using a single frequency component, so that

$$Z_t'\gamma \cong \alpha(t) = \gamma_1 \sin(2\pi kt/T) + \gamma_2 \cos(2\pi kt/T) \tag{2}$$

where: k represents the single frequency selected for the approximation, and γ_1 and γ_2 measure the amplitude and displacement of the frequency component.

The standard linear specification emerges as a special case of (1) obtained by setting $\gamma_1 = \gamma_2 = 0$. However, if there is a structural break, at least one frequency component must be present in the data-generating process. It should be clear that (2) can best mimic a break when $\alpha(t)$ is smooth. Nevertheless, consider the solid line in Panel 1 of Figure 1 that depicts a sharp structural break in the intercept of a series $\{y_t\}$. The break is such that $y_t = 0.80$ for $t \leq 33$ and $t > 66$ and $y_t = 1.0$ otherwise. The dashed line in the panel shows the approximation to the series obtained by setting $k = 1$, $\gamma_1 = 0.004$, $\gamma_2 = -0.112$, $X_t = [1]$, and $\beta = 0.868$. The values of β, γ_1 and γ_2 were selected by regressing y_t on Z_t and a constant for each integer frequency in the interval (1, 5). The frequency $k = 1$ was selected as it provided the smallest value of the sum of squared residuals ($SSR = 0.275$). In contrast, if we use only an intercept term, $SSR = 0.898$. Panel 2 shows the effect of moving the break towards the beginning of the sample. It is interesting to note that the fit of the approximation used in Panel 2 is identical to that of Panel 1 ($SSR = 0.275$) although the values of β, γ_1, and γ_2 differ. For a given size and duration, the location of the break will not affect the sum of squared residuals obtained by regressing a single frequency component on the series in question. This observation is important since some tests for breaks, such as the Bai and Perron (1998) test, have little power to detect u-shaped breaks or breaks located near the end of a series.

Panels 3 and 4 show multiple breaks and Panels 5 through 9 show breaks in the intercept and/or slope of trending functions. In these last five cases, the dashed lines were obtained by setting $X_t = [1, t]'$. For our purposes, the precise parameter values used in the approximations are not especially important. The key point to note is that a Fourier approximation of a structural break using a single integer frequency can often mimic the pattern in the data reasonably well. Since breaks shift the spectral density function towards frequency zero, the search for the most appropriate frequency k can occur at the low end of the spectrum.

Although the approximation seems to work quite well, our aim is not to develop another test for a break or to explicitly model the form of the break(s). In fact, if the nature of the breaks is known, it might be preferable to use an alternative specification for Z_t. Instead, our goal is to develop a test for stationarity that treats the breaks as unknown nuisance parameters. We use the fact that $Z_t'\gamma$ can mimic various nonlinearities as the basis for the stationarity test. Below, we show that the test for $\sigma_u^2 = 0$ is invariant to the values of β, γ_1 and γ_2 present in the data-generating process. As such, it becomes possible to test for stationarity in the presence of otherwise neglected breaks or nonlinearities.

When Z_t is absent, the DGP in (1) corresponds to that of KPSS. They suggest testing the null hypothesis of stationarity (i.e., $\sigma_\mu^2 = 0$) using the sample statistic

$$\tau_{\text{KPSS}} = \frac{1}{T^2} \frac{\sum_{t=1}^{T} \hat{S}_t^2}{\hat{\sigma}^2}$$

where $\hat{S}_t = \sum_{j=1}^{t} \hat{e}_j$ and $\hat{e}_t$ are the OLS residuals obtained from regressing y_t on X_t. The estimate of the long-run error variance, $\sigma^2 = \lim_{T\to\infty} T^{-1} E(S_T^2)$, where $S_T = \sum_{t=1}^{T} \varepsilon_t$, can be obtained using a nonparametric correction for non-*iid* errors. We need only a slight modification of the KPSS statistic when allowing for the existence of a time-varying intercept under the null hypothesis. Simply, let $\hat{e}_t$ denote the residuals from the following regression:

$$y_t = \alpha + \gamma_1 \sin(2\pi kt/T) + \gamma_2 \cos(2\pi kt/T) + e_t \tag{3a}$$

or

$$y_t = \alpha + \beta t + \gamma_1 \sin(2\pi kt/T) + \gamma_2 \cos(2\pi kt/T) + e_t \tag{3b}$$

As such, it is possible obtain the following test statistics:

$$\tau_\mu(k) \text{ or } \tau_\tau(k) = \frac{1}{T^2}\frac{\sum_{t=1}^{T} \tilde{S}_t(k)^2}{\tilde{\sigma}^2} \tag{4}$$

where $\tilde{S}_t(k) = \sum_{j=1}^{t} \tilde{e}_j$ and $\tilde{e}_j$ are the OLS residuals from the regression (3a) for $\tau_\mu(k)$ or (3b) for $\tau_\tau(k)$. As in KPSS, a nonparametric estimate $\tilde{\sigma}^2$ of the long-run variance can be obtained by choosing a truncation lag parameter l and a set of weights w_j, $j = 1,..,l$: $\tilde{\sigma}^2 = \tilde{\gamma}_0 + 2\Sigma w_j \tilde{\gamma}_j$, where $\tilde{\gamma}_j$ is the j^{th} sample autocovariance of the residuals from (3a) or (3b). In the Appendix, it is shown that the residuals and, consequently, the test statistics are dependent on the chosen frequency k. As we are not especially interested in the form of the breaks, and given that breaks are 'low frequency' events, it is reasonable for the applied researched to select a value of $k = 1$ (or possibly 2) so as to mimic the essential features of the unknown breaks.

To obtain the asymptotic distribution of our test statistics, $\tau_\mu(k)$ or $\tau_\tau(k)$, we need the following results, where we let $[rT]$, $r \in [0, 1]$, be an integer close to rT.

Proposition 1

(a) $\frac{1}{T}\sum_{t=1}^{T} \sin(2\pi kj/T) \rightarrow \frac{1}{2\pi k}\left(1-\cos(2\pi k)\right) \equiv s_0$

(b) $\frac{1}{T}\sum_{t=1}^{T}\cos(2\pi kj/T) \rightarrow \frac{1}{2\pi k}\sin(2\pi k) \equiv c_0$

(c) $\frac{1}{T}\sum_{j=1}^{[rT]}\sin(2\pi kj/T) \rightarrow \frac{r}{2\pi k}\left(1-\cos(2\pi kr)\right) \equiv s_r$

(d) $\frac{1}{T}\sum_{j=1}^{[rT]}\cos(2\pi kj/T) \rightarrow \frac{r}{2\pi k}\sin(2\pi kr) \equiv c_r$

(e) $\frac{1}{T^2}\sum_{t=1}^{T} t\cdot\sin(2\pi kj/T) \rightarrow \frac{1}{(2\pi k)^2}\sin(2\pi k)-\frac{1}{2\pi k}\cos(2\pi k) \equiv s_1$

(f) $\frac{1}{T^2}\sum_{t=1}^{T} t\cdot\cos(2\pi kj/T) \rightarrow \frac{1}{(2\pi k)^2}\left[\cos(2\pi k)+(2\pi k)\sin(2\pi k)-1\right] \equiv c_1$

(g) $\frac{1}{T}\sum_{t=1}^{T}\sin^2(2\pi kt/T) \rightarrow \frac{1}{2}-\frac{\sin(4\pi k)}{4\pi k} \equiv s_2$

(h) $\frac{1}{T}\sum_{t=1}^{T}\cos^2(2\pi kt/T) \rightarrow \frac{1}{2}+\frac{\sin(4\pi k)}{4\pi k} \equiv c_2$

(i) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_t \rightarrow \sigma W(1) \equiv \sigma f_1$

(j) $\frac{1}{T^{1.5}}\sum_{t=1}^{T} t\cdot e_t \rightarrow \sigma[W(1)-\int_0^1 W(r)dr] \equiv \sigma f_2$

(k) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_t\sin(2\pi kt/T) \rightarrow \sigma(2\pi k)[W(1)+(2\pi k)\int_0^1 \sin(2\pi kr)W(r)dr] \equiv \sigma f_3$

(l) $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_t\cos(2\pi kt/T) \rightarrow \sigma(2\pi k)^2[\int_0^1 \cos(2\pi kr)W(r)dr] \equiv \sigma f_4$

In cases where k is restricted to be an integer, it is possible to simplify terms such that: $s_0 = 0$, $c_0 = 0$, $s_1 = -1/(2\pi k)$, $c_1 = 0$, $s_2 = 1/2$ and $c_2 = 1/2$. Utilizing the above results, we can show that the asymptotic distribution of $\tau_i(k)$, $i = \mu, \tau$, is given as follows.

Theorem 1: *Suppose that y_t is generated by the DGP in* (1) *with* $\sigma_u^2 = 0$ *under the null, and one adopts the testing regression* (*3a*) *or* (*3b*). *Then, under the null hypothesis:*

$$\tau_i(k) \rightarrow \int_0^1 \underline{V}_i(r)^2 dr,\ i = \mu,\ \tau \tag{5}$$

where $\underline{V}_i(r)$ *is the projection of the Wiener process* $W(r)$ *on the orthogonal complement of the space spanned by the nonlinear function* $f_i(k,r)$, $i = \mu,\ \tau$, with $f_\mu(k,r) = [1,\ \sin(2\pi kr),$

$\cos(2\pi kr)]'$ *and* $f_\tau(k,r) = [1, r, \sin(2\pi kr), \cos(2\pi kr)]'$*, as defined over the interval* $r \in [0,1]$*, such that* $\underline{V}_i(r) = W(r) - f_i(k,r)\hat{\gamma}$*, with* $\hat{\gamma} = argmin \int_0^1 (W(r) - f_i(k,r)\gamma)^2 dr$*. The precise expressions of* $\underline{V}_i(r)$*,* $i = \mu, \tau$*, are relegated in* (*A.*2) *and* (*A.4*) *in the Appendix.*

It is clear that the asymptotic distributions of the resulting test statistics depend on the frequency k, but are invariant to the other parameters in the DGP. To obtain critical values via simulations, we employ the DGP in (1) with $\beta = \gamma = 0$. Pseudo-*iid* $N(0,1)$ random numbers were generated using the Gauss procedure RNDN and all calculations were conducted using the Gauss software version 6.0.10. The initial values y_0 and ε_0 are assumed to be random, and we set $\sigma_\varepsilon^2 = 1$. The critical values are reported in Table 1a for the sample sizes $T = 100$ and 500. The critical values were calculated using 50,000 Monte Carlo replications for values of $k = 1,\ldots, 10$.

As illustrated in Figure 1, selecting the value $k =1$ (or possibly $k = 2$) is sufficient to replicate the essential details of a large number of breaks. Given this pre-specified value of k, it is possible to estimate equation (3a) or (3b), calculate the value of $\tau_\mu(k)$ or $\tau_\tau(k)$, and perform the stationarity test using the critical values in Table 1a. We do not recommend using higher frequencies since Becker, Enders and Hurn (2004) show that the higher frequencies are likely to be associated with stochastic parameter variability (not structural breaks). Moreover, the low frequencies are the most likely ones to interfere with a test for stationarity versus nonstationarity. We illustrate this point in Section 4 using real exchange data.

One relevant issue concerns the effect of ignoring a break or nonlinear trend when using the standard KPSS test. Perron (1989) suggested that there is a bias against rejecting a false unit root if an existing structural break is ignored in the usual DF test. The nature of the problem is different in stationarity tests since the null and alternative hypotheses are reversed. Lee *et al.*

(1997) initially pointed out that stationarity tests exhibit size distortions under the null rather than a loss of power. We observe a similar problem when an existing nonlinear term is ignored. Formally, we can show

Lemma 1: *Suppose that a nonlinear term exists in the data such that the DGP implies* (1) *with* $\gamma \neq 0$*, but the nonlinear term is ignored and usual KPSS tests are employed. Then,*

$$\tau_{KPSS} = O_p(T)$$

Proof. See the Appendix.

As such, the usual KPSS-type stationarity tests will diverge when nonlinear trends are ignored. This leads to over-rejections of the true stationarity null hypothesis in favor of the false unit root hypothesis. Thus, it is important to control for nonlinear trends in stationary tests.

2a. A Supplementary Test for Linearity

The test for the null hypothesis $\sigma_u^2 = 0$ makes no specific assumption concerning the presence of breaks in the DGP. This is possible because, as shown in Theorem 1, the test statistics for $\tau_\mu(k)$ and $\tau_\tau(k)$ are invariant of the magnitudes of γ_1 and γ_2. However, if it turns out that Z_t does not belong in the DGP, it is possible to obtain increased power by using the standard KPSS test. However, a standard F-test for the null hypothesis $\gamma_1 = \gamma_2 = 0$ may not be appropriate due to the problem that k is unidentified under the null hypothesis of linearity. Furthermore, the test is affected significantly if the error term exhibits autocorrelation. Specifically, if the $\{e_t\}$ series is not *i.i.d.*, a standard F-test tends to over-reject the correct null hypothesis of $\gamma = (0, 0)'$. In this case, one straightforward solution to the problem is to use the bootstrapping methodology as detailed in the Appendix.

2b. A Data-Driven Method of Selecting *k*

In most instances the value $k = 1$ or $k = 2$ should be sufficient to capture the important breaks in the data. In this subsection, we explore the properties of a completely data-driven method of selecting the frequency. We follow Davies (1987) by selecting the value of $k = \hat{k}$ that minimizes the sum of squared residuals from regression (3a) or (3b). Specifically, for each integer value of k in the interval $1 \leq k \leq k^{max}$, we estimate (3a) or (3b) and select the value yielding the best fit.[3] As a practical matter, we set the maximum frequency at $k^{max} = 5$ since low frequencies are associated with breaks. Now consider constructing (4) using the estimated frequency $\hat{k}$. We denote the test statistic utilizing $\hat{k}$ by:

$$\tau_i(\hat{k}) = \tau_i(k) \text{ with } \hat{k},\ i = \mu,\ \tau. \tag{6}$$

Practically speaking, the distribution of $\tau_i(\hat{k})$ depends on how accurately the selected trigonometric component mimics the essential features of the DGP. The simulation results reported in Section 3 show encouraging results that the unknown frequency k is estimated quite accurately even in small samples. As such, it seems that Davis' grid-search procedure yields a consistent estimated of k when the true DGP is given by (3a) or (3b). Thus, we suggest using the critical values in Table 1 for the estimated value of k.

Moreover, if k is estimated, a test for the null hypothesis of linearity (i.e., $\gamma_1 = \gamma_2 = 0$) is complicated by the fact that k is unidentified under the null hypothesis $\gamma_1 = \gamma_2 = 0$. As such, the distribution of the statistic for this hypothesis is non-standard. Consider the following F-test statistic that calculated against the alternative nonlinear trend with a given frequency k

$$F_i(k) = \frac{(SSR_0 - SSR_1(k))/2}{SSR_1(k)/(T-q)},\ \ i = \mu,\ \tau, \tag{7}$$

[3] Readers familiar with Davis (1987) will recognize that he develops a test for the unobserved frequency component $Z_t = \gamma_1 \sin(2\pi kt/T) + \gamma_2 \cos(2\pi kt/T)$ with *iid* errors.

Here, $SSR_1(k)$ denotes the SSR from equation (3a) or (3b), q is the number of regressors, and SSR_0 denotes the SSR from the regression without the trigonometric terms.

When k is unknown, it is not possible to use (7) directly. Instead, consider:

$$F_i(\hat{k}) = \underset{k}{Max}\, F(k),\ i = \mu,\ \tau, \tag{8}$$

where $\hat{k} = \underset{k}{argmax}\, F(k)$. It should be clear that (8) is a supremum test since $\hat{k}$ minimizes the SSR for (3a) or (3b); i.e., the F-statistic in (8) is such that $\hat{k} = \underset{k}{arginf}\, SSR_1(k)$. Since the distribution of the F-statistic is non-standard, we simulated the critical values of $F_i(\hat{k})$ and report them in Table 1b. To utilize our tests, first obtain $\hat{k}$ by minimizing the SSR and apply the F-test under the null of stationarity with $F_i(\hat{k})$ to examine whether a nonlinear trend exists. The test should be conducted using the critical values in Table 1b. If the null hypothesis of linearity is not rejected, use the usual KPSS statistics without the trigonometric terms. If linearity is rejected, use the $\tau_\mu(k)$ or $\tau_\tau(k)$ test statistics reported in Table 1a with $k = \hat{k}$. The critical values for the F-test given in Table 1b are generated under *iid* errors using the DGP (1) under the null hypothesis with $\beta = \gamma = 0$.

3. Small Sample Performance

In this section, simulation evidence is presented demonstrating that the proposed strategy has acceptable size and power properties in the sample sizes often used in applied work. All of our Monte Carlo experiments use 20,000 replications.

Our approach is a two-step procedure in that we test for nonlinearity and then test for stationarity. We estimate (3a) or (3b) and test the null hypothesis $\gamma_1 = \gamma_2 = 0$ using the supremum

F-test of equation (8). In the second step, if the null of linearity cannot be rejected at the 5% level, we apply usual KPSS tests. Otherwise, when the null of linearity is rejected, we use our test statistics, $\tau_i(\hat{k})$, $i = \mu, \tau$, reported in Table 1a. We first examine the size of the tests using the grid-search method to estimate the frequency k. Clearly, the size can be improved if the 'correct' frequency is used. Table 2 reports the 1, 5 and 10% rejection frequencies for various values of T, k, γ_1 and γ_2 in the DGP. It is evident that the size is quite good when $\gamma_1 = \gamma_2 = 1.0$ or 0.5. It is encouraging to see how often the search procedure picks the correct trigonometric frequency. In these cases, the correct frequency is picked almost 100% of the time. Not surprisingly, the percentage of correct picks increases with sample size and with the importance of the trigonometric terms in the DGP. In Table 2, we also report the percentages of under-estimating ($\hat{k} < k$), over-estimating ($\hat{k} > k$) and correctly estimating ($\hat{k} = k$) the unknown trigonometric frequency k. An examination of the table shows that the unknown frequency is fairly well estimated. Only when there is weak evidence such that the coefficients are small along with a small sample size ($T = 100$, and $\gamma_1 = \gamma_2 = 0.2$), the testing procedure tends to be too conservative and the trigonometric terms are not clearly detected. In this case, the frequency of detecting linearity increases, and the tests exhibit only mild size distortion. When the sample size increases, the chance of detecting correct values of k accordingly increases.

Lemma 1 shows that the usual KPSS type stationarity tests will diverge when unattended breaks are ignored. We now examine the magnitude of the size distortion in finite samples. In this experiment, the DGP implies (1) such that nonlinearity exists in the data, but the usual KPSS tests are employed. The simulation results are shown in Table 3. It is instructive to compare these size properties against the size properties of the standard KPSS test when neglecting the presence of the trigonometric terms. While it is not surprising that the standard KPSS test is clearly oversized when the amplitude of the trigonometric terms is high, it is important to note the significant size distortion for very small trigonometric amplitudes or for a large sample size.

We next examine the power properties of our tests. A concern is that the inclusion of trigonometric functions masks the nonstationarity of a time series. Since a nonstationary time series can move in long swings, the inclusion of a low frequency trigonometric component might result in a non-rejection of stationarity. Nevertheless, our Monte Carlo experiments demonstrate that this potential problem is rather mild. The DGP under consideration is equation (1) such that $\sigma_u^2 = \lambda$ and $\varepsilon_t \sim N(0, 1)$. Thus, the magnitude of λ will determine the strength of the departure from the null hypothesis. Table 4 reports the simulation results using a known frequency of k = 1, 2, 3 or 5 as well as using estimated frequency ($\hat{k}$). For comparison purposes, the simulation results on the power of the usual KPSS tests (not allowing for a nonlinear trend) are reported as well. As shown in the table, the power properties of our $\tau_\mu(k)$ and $\tau_\tau(k)$ tests using 5% critical values are largely comparable to those of the KPSS tests. When λ is small, say $\lambda < 0.001$ or lower, the DGP is essentially the same as the case where $\sigma_u^2 \approx 0$. When λ gets bigger ($\sigma_u^2 > 0$), the rejection frequencies increase. Note that the power of the test is quite good even when using the estimated frequency $\hat{k}$.

4. Real Exchange Rates and Smooth Structural Breaks

As judged by the number of papers on the topic, one of the more contentious issues in the economics literature concerns the validity of purchasing power parity (*ppp*) during the post-Bretton Woods period. At the time of this writing, *Econlit* contains 453 papers with the term 'purchasing power parity' in the title and 896 papers with *ppp* in the title and/or the abstract. Part of the reason for the unresolved ambiguity is that real exchange rates are highly persistent and unit-root tests have low power over relatively short time spans. Moreover, Perron and Vogelsang (1992) added another dimension to the problem by considering the possibility that real exchange rates contain structural breaks. As such, it seems natural to test *ppp* under the null of stationarity allowing for unknown structural breaks.

In order to illustrate our test, we obtained quarterly values of the Canadian, Japanese and U.K. nominal exchange rates against the U.S. dollar from the CD-ROM version of the *International Financial Statistics.* We also collected producer and consumer price indices and for each country formed the real exchange rate as:

$$r_t = e_t + p_t - p_t^*$$

where r_t = real exchange rate against the U.S. dollar, p_t = U.S. price level, p_t^* = foreign price level, e_t = foreign currency price of the U.S. dollar, and all variables are expressed in natural logarithms such that $r_{1996:1} = 0$.

The time paths of the three real exchange rates constructed using the producer price indices are shown as the solid lines in the left-hand column of Figure 2. The three rates exchange rates constructed using the consumer price indices are shown as the solid lines in the right-hand column of Figure 2. Although all of the real rates experienced a prolonged downward movement beginning in 1985, it is not clear whether this is the result of a break or a unit root in the data generating process. Moreover, 1976 might be the beginning of a break for the Canadian rates and 1979 might serve as the beginning of a break for the U.K. rates.

To compare our test to the standard linear unit-root tests, we first estimated an augmented Dickey-Fuller equation for each real exchange rate series. Consider

$$\Delta y_t = \alpha_0 + \rho y_{t-1} + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \varepsilon_t$$

We excluded *time* as a regressor since any type of trend component is inconsistent with *ppp*. The estimated value of ρ, the lag length (p), and the t-statistic for the null hypothesis $\rho = 0$ are reported in the third through the fifth columns of Table 5. At the 5% significance level, it is possible to reject the null hypothesis of a unit root only for the U.K./U.S. real exchange rate constructed using consumer price indices.

Next, we applied a standard KPSS test to each series. Since there is a substantial amount of persistence in each series, we experimented with various truncation lags when estimating the long-run variances. When we used an 8-quarter truncation lag for all series, the KPSS-statistics reported in Table 5 are such that the null hypothesis of stationarity is maintained for the Canadian/U.S. rate constructed with producer prices and the U.K./U.S. real exchange rate constructed using consumer price indices. The results are virtually identical when using the optimal truncation lag, as in Andrew's (1991) for each series, individually.

The situation is quite different when we applied our nonlinear version of the stationarity test to the data. Although our recommendation is to use $k = 1$ or 2, we report results using the grid-search method described in Section 2. First, as in (3a), we regressed each real rate on a constant, $\sin(2\pi kt/T)$ and $\cos(2\pi kt/T)$ for each integer frequency in the interval (1, 5). The frequencies resulting in the best fit ($\hat{k}$) are reported in Table 5. It is interesting to note that the sole value of $\hat{k}$ greater than 2 is that for the real U.K./U.S. rate constructed using CPIs. However, this is the single case found to be stationary by the DF and the standard KPSS test. The suggestion is that it is more important to control for neglected low frequency components in standard DF and KPSS tests.

The F-statistics for the null hypothesis $\gamma_1 = \gamma_2 = 0$ are shown in the table. Since these coefficients depend on the nuisance parameter k, it is necessary to perform the test using the critical values reported in Table 1b. The estimated time paths of the time-varying intercepts are shown by the dashed lines in Figure 2. An examination of the figure indicates that the trigonometric approximations all seem reasonable and all support the notion of long swings in real exchange rates.

The key issue, however, is to determine whether the various series are stationary. As such, we calculated $\tau_\mu(\hat{k})$ as in (6) using the estimated frequencies. We used various methods to select the truncation lag for constructing the long-run variances. A common value of 8 worked best for the series as a group and the results are not very sensitive for the individual lags selected by Andrew's (1991) automatic bandwidth selection method. At the 5% significant level, Table 1 indicates that the appropriate critical values are 0.1724, 0.4113 and 0.4477 for the frequencies k = 1, 2, and 3, respectively. Comparing these values to the calculated sample values reported in Table 5 indicates that the null hypothesis of stationary can be rejected only for the Canadian/U.S. rate constructed using consumer prices.

5. Summary and Concluding Remarks

In many circumstances, it is natural to test the validity of a theory such that the theory itself forms the basis of the null hypothesis and its refutation forms the alternative. Standard unit root tests, such as the Dickey-Fuller test, have the null hypothesis of a unit root and the alternative hypothesis of stationarity. For this reason, KPSS-type tests have become popular for testing theories, such as purchasing power parity, that imply that a particular variable is stationary. In the recent time-series literature, several papers have extended the KPSS-type tests to allow for one sharp break. Nevertheless, many macroeconomic time-series variables are subject to an unknown number of breaks with unknown functional forms, and such structural changes can follow a smooth transition process. We develop a stationarity test for such circumstances that relies on the fact that a single frequency component of a Fourier approximation can mimic a wide variety of breaks and other types of nonlinearities. Instead of selecting specific break dates, the number of breaks, and the form of the breaks, we suggest including a single frequency component to include in the estimating equation. Since breaks shift the spectral density function toward zero, we suggest controlling for breaks using a frequency of 1 or 2.

When the most appropriate frequency component is unknown, we use a grid-search method over the low frequencies to find a consistent estimate of the frequency. The frequency component itself is treated as a nuisance parameter since our ultimate aim is to develop a test for stationarity. The asymptotic distribution for our test is derived and the empirical performance is such that the test does not exhibit any serious size distortions, and shows reasonable power. The appropriate use of the test is illustrated using real exchange rates in the post-Bretton Woods period. The theory of purchasing power parity implies that such rates should be stationary. However, a standard Dickey-Fuller test allows us to reject a unit root for only one of the six rates considered. Instead, our KPSS-type test allowing for multiple breaks supports *ppp* for five of the six real exchange rates.

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Table 1a

Critical Values for $\tau_\mu(k)$ and $\tau_\tau(k)$

T	k	$\tau_\mu(k)$ – Level			$\tau_\tau(k)$ – Trend		
		10%	5%	1%	10%	5%	1%
100	1	0.1319	0.1724	0.2711	0.0471	0.0546	0.0713
	2	0.3120	0.4113	0.6470	0.1018	0.1309	0.2000
	3	0.3401	0.4477	0.7157	0.1147	0.1437	0.2167
	5	0.3513	0.4625	0.7320	0.1207	0.1494	0.2163
500	1	0.1297	0.1710	0.2703	0.0466	0.0545	0.0739
	2	0.3038	0.4076	0.6539	0.0990	0.1278	0.1969
	3	0.3309	0.4425	0.7140	0.1118	0.1403	0.2077
	5	0.3402	0.4488	0.7309	0.1164	0.1461	0.2163

Table 1b

Critical Values for $F_i(\hat{k})$

T	$F_\mu(\hat{k})$ – Level			$F_\tau(\hat{k})$ – Trend		
	10%	5%	1%	10%	5%	1%
100	4.133	4.929	6.730	4.162	4.972	6.873
500	3.935	4.651	6.281	3.928	4.669	6.315

Table 2

Rejection frequencies of $\tau_\mu(\hat{k})$ and $\tau_\tau(\hat{k})$ tests under the null

T	k	$\tau_\mu(\hat{k})$ – Level $\hat{k}<k$	$\hat{k}=k$	$\hat{k}>k$	1%	5%	10%	$\tau_\tau(\hat{k})$ – Trend $\hat{k}<k$	$\hat{k}=k$	$\hat{k}>k$	1%	5%	10%
		$\gamma_1=\gamma_2=1$											
100	1	0%	100%	0%	0.011	0.049	0.101	0%	100%	0%	0.010	0.048	0.097
	2	0%	100%	0%	0.012	0.050	0.102	0%	100%	0%	0.012	0.053	0.102
	3	0%	100%	0%	0.012	0.051	0.100	0%	100%	0%	0.009	0.048	0.093
	5	0%	100%	0%	0.009	0.049	0.099	0%	100%	0%	0.010	0.050	0.100
500	1	0%	100%	0%	0.009	0.048	0.097	0%	100%	0%	0.010	0.052	0.103
	2	0%	100%	0%	0.009	0.049	0.099	0%	100%	0%	0.010	0.049	0.102
	3	0%	100%	0%	0.011	0.048	0.099	0%	100%	0%	0.011	0.050	0.102
	5	0%	100%	0%	0.010	0.049	0.099	0%	100%	0%	0.011	0.047	0.096
		$\gamma_1=\gamma_2=0.5$											
100	1	0%	100%	0%	0.011	0.049	0.102	1%	97%	2%	0.012	0.052	0.102
	2	0%	100%	0%	0.012	0.052	0.102	1%	99%	0%	0.012	0.053	0.101
	3	0%	100%	0%	0.011	0.050	0.099	0%	99%	0%	0.009	0.047	0.092
	5	0%	100%	0%	0.009	0.050	0.099	0%	100%	0%	0.009	0.050	0.099
500	1	0%	100%	0%	0.009	0.048	0.097	0%	100%	0%	0.010	0.052	0.103
	2	0%	100%	0%	0.009	0.049	0.099	0%	100%	0%	0.010	0.049	0.102
	3	0%	100%	0%	0.011	0.048	0.099	0%	100%	0%	0.011	0.050	0.102
	5	0%	100%	0%	0.010	0.049	0.099	0%	100%	0%	0.011	0.047	0.096
		$\gamma_1=\gamma_2=0.2$											
100	1	32%	47%	21%	0.009	0.049	0.107	39%	36%	25%	0.004	0.033	0.100
	2	37%	47%	15%	0.005	0.035	0.078	39%	45%	17%	0.004	0.027	0.065
	3	42%	48%	10%	0.003	0.032	0.071	43%	47%	11%	0.002	0.019	0.053
	5	52%	48%	0%	0.004	0.031	0.070	52%	48%	0%	0.003	0.021	0.054
500	1	0%	99%	1%	0.009	0.049	0.099	2%	93%	5%	0.011	0.056	0.111
	2	1%	99%	1%	0.009	0.049	0.098	1%	98%	1%	0.010	0.048	0.101
	3	1%	99%	1%	0.010	0.047	0.098	1%	98%	1%	0.010	0.047	0.098
	5	1%	99%	0%	0.009	0.047	0.098	1%	99%	0%	0.012	0.045	0.093

Note: DGP implies (1) where trig functions are present. The frequency is selected according to the SSR minimization criterion.

Table 3
Rejection frequencies for standard KPSS tests
when trig functions are present but ignored

T	k	$\gamma_1 = \gamma_2 = 1.0$			$\gamma_1 = \gamma_2 = 0.5$			$\gamma_1 = \gamma_2 = 0.2$		
		1%	5%	10%	1%	5%	10%	1%	5%	10%
Level KPSS										
100	1	1.000	1.000	1.000	0.784	0.951	0.986	0.089	0.255	0.386
	2	0.337	0.840	0.986	0.070	0.244	0.421	0.017	0.082	0.150
	3	0.027	0.180	0.360	0.019	0.092	0.175	0.012	0.056	0.117
	5	0.002	0.023	0.067	0.006	0.039	0.086	0.008	0.046	0.099
500	1	1.000	1.000	1.000	1.000	1.000	1.000	0.740	0.918	0.965
	2	1.000	1.000	1.000	0.971	1.000	1.000	0.099	0.289	0.462
	3	1.000	1.000	1.000	0.334	0.753	0.951	0.034	0.126	0.227
	5	0.185	0.632	0.907	0.045	0.181	0.325	0.015	0.065	0.131
Trend KPSS										
100	1	1.000	1.000	1.000	0.896	0.972	0.986	0.135	0.321	0.442
	2	1.000	1.000	1.000	0.297	0.762	0.894	0.021	0.114	0.221
	3	0.216	0.960	0.998	0.023	0.197	0.421	0.011	0.064	0.128
	5	0.001	0.033	0.113	0.006	0.047	0.106	0.009	0.052	0.104
500	1	1.000	1.000	1.000	1.000	1.000	1.000	0.801	0.929	0.963
	2	1.000	1.000	1.000	1.000	1.000	1.000	0.331	0.713	0.853
	3	1.000	1.000	1.000	0.998	1.000	1.000	0.057	0.283	0.488
	5	1.000	1.000	1.000	0.136	0.704	0.943	0.015	0.088	0.173

Table 4

5% Rejection frequencies of $\tau_\mu(k)$ and $\tau_\tau(k)$ tests under the alternative

		value of λ						
T	k	0.0001	0.001	0.01	0.1	1	100	10000
		$\tau_\mu(k)$ – Level						
100	1	0.053	0.097	0.364	0.785	0.962	0.988	0.987
	2	0.057	0.154	0.552	0.917	0.985	0.989	0.991
	3	0.063	0.163	0.582	0.929	0.990	0.995	0.995
	5	0.068	0.167	0.585	0.923	0.991	0.996	0.995
	$\hat{k}$	0.031	0.032	0.039	0.335	0.943	0.974	0.974
	τ_{KPSS}	0.063	0.168	0.587	0.927	0.989	0.994	0.998
500	1	0.152	0.561	0.945	0.999	1.000	1.000	1.000
	2	0.289	0.767	0.981	0.999	1.000	1.000	1.000
	3	0.303	0.783	0.987	1.000	1.000	1.000	1.000
	5	0.305	0.797	0.987	1.000	1.000	1.000	1.000
	$\hat{k}$	0.029	0.030	0.145	0.911	0.988	0.988	0.988
	τ_{KPSS}	0.307	0.788	0.997	1.000	1.000	1.000	1.000
		$\tau_\tau(k)$ – Trend						
100	1	0.051	0.064	0.144	0.702	0.988	0.999	1.000
	2	0.058	0.080	0.311	0.788	0.966	0.987	0.987
	3	0.053	0.082	0.346	0.860	0.985	0.997	0.997
	5	0.049	0.082	0.344	0.879	0.992	0.999	0.998
	$\hat{k}$	0.023	0.026	0.026	0.313	0.943	0.974	0.974
	τ_{KPSS}	0.054	0.084	0.352	0.878	0.993	0.999	0.999
500	1	0.072	0.329	0.954	1.000	1.000	1.000	1.000
	2	0.137	0.530	0.933	0.999	1.000	1.000	1.000
	3	0.137	0.607	0.976	1.000	1.000	1.000	1.000
	5	0.142	0.606	0.982	1.000	1.000	1.000	1.000
	$\hat{k}$	0.020	0.020	0.115	0.911	0.988	0.988	0.988
	τ_{KPSS}	0.137	0.621	0.983	1.000	1.000	1.000	1.000

Table 5
Tests for Unit Roots and Stationarity in the Real Exchange Rate Series

	DF test			KPSS	Nonlinear Stationarity test		
	lags	ρ	τ	τ_{KPSS}	$\hat{k}$	$F_{\mu}(\hat{k})$	$\tau_{\mu}(\hat{k})$
Producer Prices							
Canada	3	–0.052	–2.11	0.365*	2	70.01	0.196*
U.K.	7	–0.068	–2.20	0.921	1	61.55	0.127*
Japan	3	–0.054	–2.12	0.700	1	95.94	0.048*
Consumer Prices							
Canada	3	–0.020	–1.64	0.935	2	66.52	0.958
U.K.	7	–0.138	–3.22*	0.281*	3	41.76	0.245*
Japan	3	–0.050	–2.21	0.837	1	90.59	0.073*

Note: Lags is the number of lags selected for the augmented Dickey-Fuller test, ρ is the coefficient of interest in the Dickey-Fuller test, τ is the DF t-statistic for the null hypothesis $\rho = 0$, and τ_{KPSS} is the value of the standard KPSS test for the null hypothesis of stationarity. At the 5% significance level, the critical value for τ is –2.89 and the critical value for KPSS is 0.463. Note that * denotes that purchasing power parity is supported at the 5% significance level.

Table 6

Significance of $F_i(\hat{k})$ evaluated using the recursive bootstrap

		Persistence parameter ρ				
T	**sig level**	0.0	0.2	0.5	0.7	0.9
100	0.01	0.0120	0.0020	0.0080	0.0080	0.0130
	0.05	0.0420	0.0400	0.0420	0.0480	0.0540
	0.10	0.0840	0.0900	0.0880	0.1020	0.1330
500	0.01	0.0200	0.0120	0.0140	0.0120	0.0170
	0.05	0.0560	0.0510	0.0500	0.0550	0.0520
	0.10	0.1010	0.0970	0.0980	0.1000	0.1060

Note: The table displays the size properties of the $F_i(\hat{k})$ test, when its significance is evaluated by means of the recursive bootstrap. The temporal dependence in the residuals of the DGP under the null is AR(1) and the bootstrap procedure estimates an AR(1) process from which to simulate bootstraps.

Figure 1: Nine Fourier Approximations

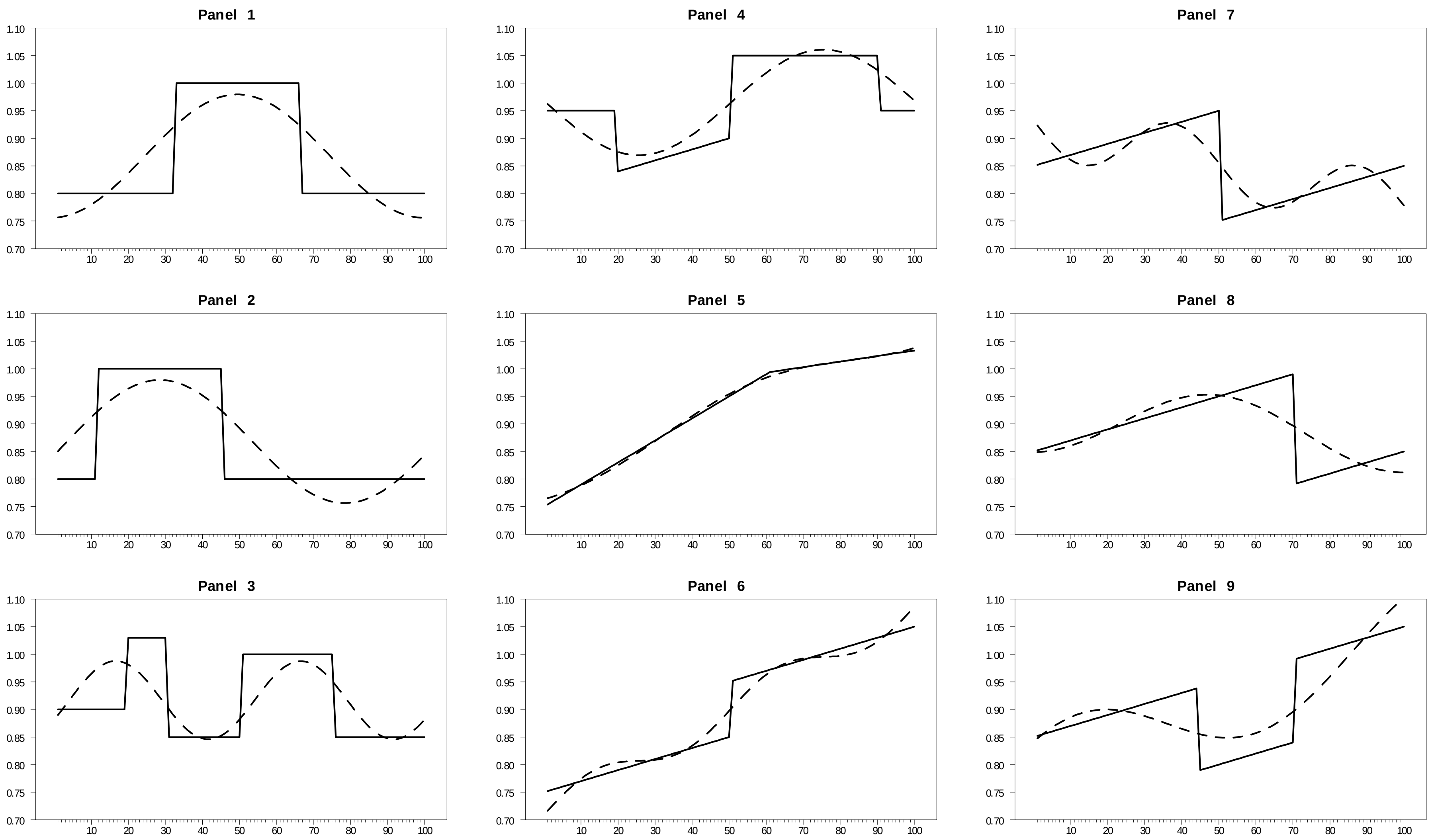
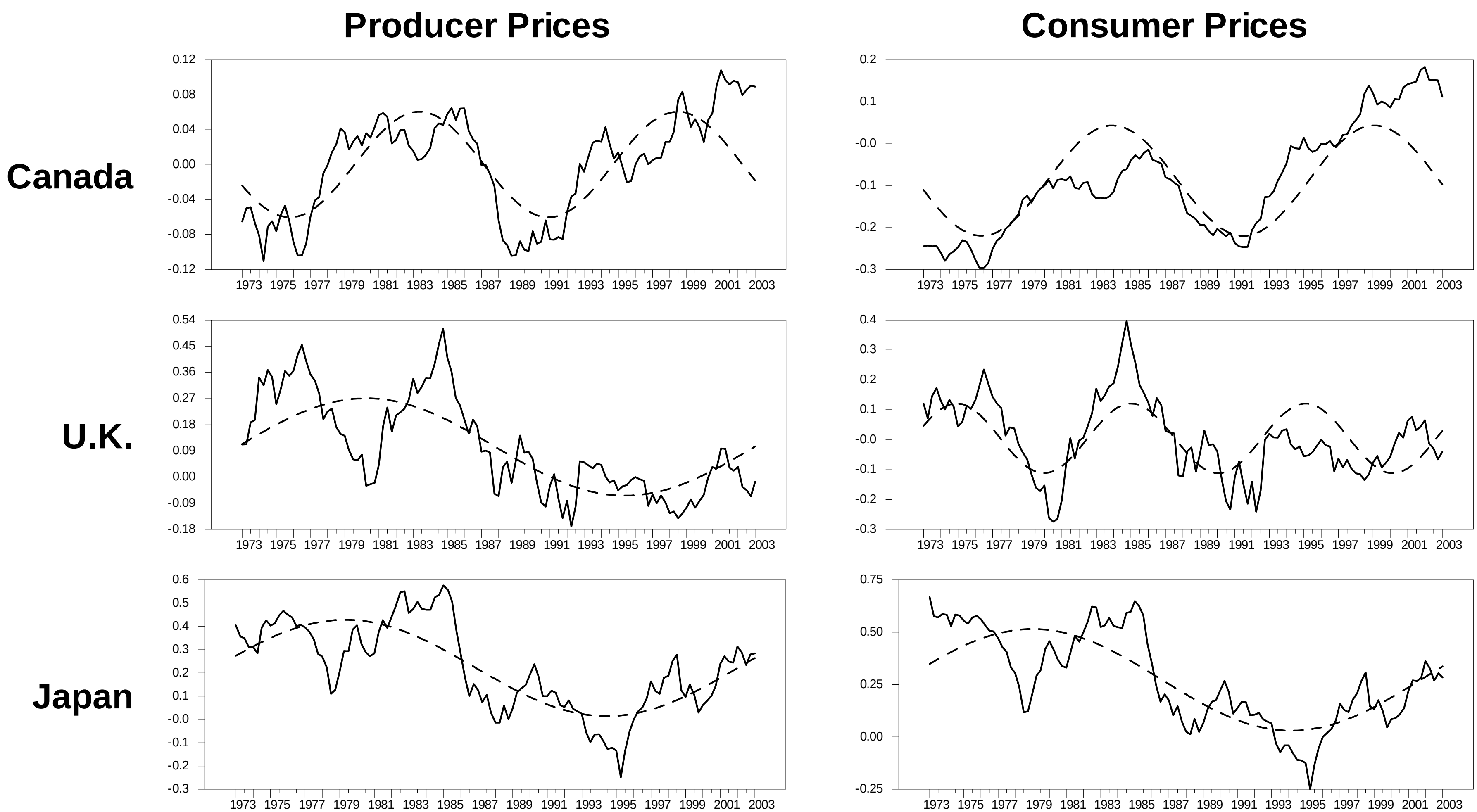


Figure 2: Logs of Real Exange Rates and Fitted Nonlinearities

(log 1996=0.0)



Appendix

Proof of Proposition 1

(a) $\frac{1}{T}\sum_{t=1}^{T}\sin(2\pi kj/T) \to \int_0^1 \sin(2\pi ka)da = \frac{1}{2\pi k}\left(1-\cos(2\pi k)\right)$

(b) $\frac{1}{T}\sum_{t=1}^{T}\cos(2\pi kj/T) \to \int_0^1 \cos(2\pi ka)da = \frac{\sin(2\pi k)}{2\pi k}$

(c) $\frac{1}{T}\sum_{j=1}^{[rT]}\sin(2\pi kj/T) = \frac{r}{rT}\sum_{j=1}^{[rT]}\sin(2\pi kj/T) \to r\int_0^r \sin(2\pi ka)da = \frac{r}{2\pi k}\left(1-\cos(2\pi kr)\right)$

(d) $\frac{1}{T}\sum_{j=1}^{[rT]}\cos(2\pi kj/T) = \frac{r}{rT}\sum_{j=1}^{[rT]}\cos(2\pi kj/T) \to r\int_0^r \cos(2\pi ka)da = r\frac{\sin(2\pi kr)}{2\pi k}$

(e) $\frac{1}{T^2}\sum_{t=1}^{T} t\cdot\sin(2\pi kj/T) \to \int_0^1 r\sin(2\pi kr)dr = \frac{1}{(2\pi k)^2}\sin(2\pi k) - \frac{1}{2\pi k}\cos(2\pi k)$

(f) $\frac{1}{T^2}\sum_{t=1}^{T} t\cdot\cos(2\pi kj/T) \to \int_0^1 r\cos(2\pi kr)dr = \frac{1}{(2\pi k)^2}\left[\cos(2\pi k) + 2\pi k\sin(2\pi k) - 1\right]$

(g) $\frac{1}{T}\sum_{t=1}^{T}\sin^2(2\pi kt/T) \to \int_0^1 \sin^2(2\pi kr)dr = \frac{1}{2}\int_0^1\left(1-\cos(4\pi kr)\right)dr = \frac{1}{2} - \frac{\sin(4\pi k)}{4\pi k}$

(h) $\frac{1}{T}\sum_{t=1}^{T}\cos^2(2\pi kt/T) \to \int_0^1 \cos^2(2\pi kr)dr = \int_0^1\left(1-\sin^2(2\pi kr)\right)dr = \frac{1}{2} + \frac{\sin(4\pi k)}{4\pi k}$

The results for (i) and (j) are standard. For the proofs for (k) and (l), we employ the result in Bierens (1994, Lemma 9.6.3): $\sum_{t=2}^{T} F(t/T)u_t = F(1)S_T(1) - \int_0^1 f(r)S_T(r)dr$, where $f(r)$ is $F'(r)$. To obtain the result for (k), we use $F(x) = \cos(2\pi kt/T)$. Then, it is easy to show that $F(1)S_T(1) - \int_0^1 f(r)S_T(r)dr = \sigma[W(1) + (2\pi k)\int_0^1 \sin(2\pi kr)W(r)dr]$. For the result in (l), we choose $F(x) = \sin(2\pi kt/T)$, and the desired result is obtained.

Proof of Theorem 1

We first consider the level stationarity test with a Fourier function. As in the text, we consider $\tau_\mu(k)$ where $\tilde{e}_t$ are the OLS residuals from the regression (3a) with $X_t = [1, \sin(2\pi kt/T), \cos(2\pi kt/T)]'$. We examine

$$\frac{1}{\sqrt{T}}\tilde{S}_{[rT]} = \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}\tilde{e}_j = \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}[e_j - X_j'D_T[D_T X'X D_T]^{-1}D_T X'e]$$

$$= \frac{1}{\sqrt{T}}S_{[rT]} - \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}X_j'D_T\cdot[D_T X'X D_T]^{-1}\cdot D_T X'e \qquad \text{(A.1)}$$

where $X = (X_1, .., X_T)'$, $e = (e_1, .., u_T)'$, $D_T = diag[1/\sqrt{T}, 1/\sqrt{T}, 1/\sqrt{T}]$; and $S_{[rT]} = \sum_{j=1}^{[rT]} e_j$. We can show

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}X_j'D_T = [r, \frac{1}{T}\sum_{j=1}^{[rT]}\sin(2\pi kj/T), \frac{1}{T}\sum_{j=1}^{[rT]}\cos(2\pi kj/T)]' \to [r, s_r, c_r]'.$$

$$[D_T X'X D_T]^{-1} = \begin{bmatrix} T/T & T^{-1}\sum\sin(2\pi kt/T) & T^{-1}\sum\cos(2\pi kt/T) \\ & T^{-1}\sum\sin^2(2\pi kt/T) & 0 \\ & & T^{-1}\sum\cos^2(2\pi kt/T)\end{bmatrix}^{-1}$$

$$\to \begin{bmatrix} 1 & s_0 & c_0 \\ & s_2 & 0 \\ & & c_2 \end{bmatrix}^{-1}$$

$$D_T X'e = [\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_t, \frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_t\sin(2\pi kt/T), \frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_t\cos(2\pi kt/T)]'$$

$$\to \sigma[f_1, f_3, f_4]'$$

Then, after a tedious algebra, we can show that the second term in (A.1) follows

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}X_j'D_T\cdot[D_T X'X D_T]^{-1}\cdot D_T X'e$$

$$\to [-c_0 c_r s_0 f_3 + rc_2 s_0 f_3 + c_r s_0^2 f_1 + c_0 c_r s_2 f_1 - c_r s_2 f_4 - rc_2 s_2 f_1 + rc_0 s_2 f_4 + c_0^2 s_r f_3$$

$$- \quad c_2 s_r f_3 + c_2 s_0 s_r f_1 - c_0 s_0 s_r f_4] / [c_2 s_0^2 + c_0^2 s_2 - c_2 s_2]$$

$$\equiv \sigma E_\mu(k, r)$$

Therefore, the expression in (A.1) follows

$$\frac{1}{\sqrt{T}}\tilde{S}_{[rT]} \to \sigma[W(r) - E_\mu(k,r)] \equiv \underline{V}_\mu(r) \qquad \text{(A.2)}$$

Then, it is easy to show that the numerator of $\tau_\mu(k)$ follows

$$\frac{1}{T^2}\sum_{t=1}^{T}\tilde{S}^2_{[rT]} \rightarrow \sigma^2\int_0^1 \underline{V}_\mu(r)^2 dr$$

As the denominator of $\tau_\mu(k)$ is a consistent estimator of σ^2, the desired result is obtained.

Next, we consider the trend stationarity test with a Fourier function, $\tau_t(k)$ where $\tilde{e}_t$ are the OLS residuals from the regression (3b) with $X_t = [1, t, \sin(2\pi kt/T), \cos(2\pi kt/T)]'$. We again examine the similar expression as in (A.1) with $D_T = diag[1/\sqrt{T}, 1/T^{1.5}, 1/\sqrt{T}, 1/\sqrt{T}]$.

$$\frac{1}{\sqrt{T}}\tilde{S}_{[rT]} = \frac{1}{\sqrt{T}}S_{[rT]} - \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]} X_j' D_T \cdot [D_T X'X D_T]^{-1} \cdot D_T X'e \qquad \text{(A.3)}$$

We can show

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]} X_j' D_T = [r, \frac{1}{T^2}\frac{1}{2}(rT)(rT-1), \frac{1}{T}\sum_{j=1}^{[rT]}\sin(2\pi kj/T), \frac{1}{T}\sum_{j=1}^{[rT]}\cos(2\pi kj/T)]'.$$

$$\rightarrow [r, \frac{1}{2}r^2, s_r, c_r]'.$$

$$[D_T X'X D_T]^{-1} = \begin{bmatrix} T/T & T^{-1}\sum t & T^{-1}\sum \sin(2\pi kt/T) & T^{-1}\sum\cos(2\pi kt/T) \\ & T^{-3}\sum t^2 & T^{-2}\sum t\sin(2\pi kt/T) & T^{-2}\sum t\cos(2\pi kt/T) \\ & & T^{-1}\sum\sin^2(2\pi kt/T) & 0 \\ & & & T^{-1}\sum\cos^2(2\pi kt/T) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1/2 & s_0 & c_0 \\ & 1/3 & s_1 & c_1 \\ & & s_2 & 0 \\ & & & c_2 \end{bmatrix}^{-1}$$

$$D_T X'e = [\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_t, \frac{1}{T^{1.5}}\sum_{t=1}^{T} te_t, \frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_t\sin(2\pi kt/T), \frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_t\cos(2\pi kt/T)]'$$

$$\rightarrow \sigma[f_1, f_2, f_3, f_4]'$$

Then, after a tedious algebra, we can show that the second term in (A.3) follows

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]} X_j' D_T \cdot [D_T X'X D_T]^{-1} \cdot D_T X'e$$

$$\to [-12c_1c_rs_1f_3 - 6rc_2s_1f_3 + 6r^2c_2s_1f_3 + 12\, c_rs_1^2f_4 + 12\, rc_2s_1^2f_1 - 6c_1c_rs_2f_1 + 12c_1c_rs_2f_2$$

$$- c_rs_2f_4 + 12rc_1^2s_2f_1 - 4rc_2s_2f_1 + 6rc_2s_2f_2 - 6rc_1s_2f_4 + 3r^2c_2s_2f_1 + 6r^2c_2s_2f_2 + 6r^2c_1s_2f_4$$

$$+ 12c_1^2s_rf_3 - c_2s_rf_3 - 6c_1s_1s_rf_1 + 12c_2s_1s_rf_2 - 12c_1s_1s_rf_4]/[12c_2s_1^2 + 12c_1^2s_2 - c_2\, s_2]$$

$$\equiv \sigma E_\tau(k, r)$$

We note in passing that we utilized the fact that $s_0 = c_0 = 0$ for integer k, to simplify the expression above. For integer values of k, the above expression can be further simplified by noting that

$$[D_T X'X D_T]^{-1} \to \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ & 1/3 & -1/(2\pi k) & 0 \\ & & 1/2 & 0 \\ & & & 1/2 \end{pmatrix}^{-1}$$

Therefore, the expression in (A.3) follows

$$\frac{1}{\sqrt{T}}\widetilde{S}_{[rT]} \to \sigma[W(r) - E_\tau(k,r)] \equiv \underline{V}_\tau(r) \qquad \text{(A.4)}$$

Then, similarly,

$$\frac{1}{T^2}\sum_{t=1}^{T} \widetilde{S}^2_{[rT]} \to \sigma^2 \int_0^1 \underline{V}_\tau(r)^2 dr$$

Accordingly, the desired result follows.

Proof of Lemma 1

We begin with the case of the level stationarity test. The DGP implies equation (1) with $\sigma_u^2 = 0$, $X_t = [1]$, and $Z_t = [\sin(2\pi kt/T), \cos(2\pi kt/T)]'$ such that a Fourier nonlinear term is present, but Z_t is ignored in the testing regressions where the usual KPSS regression is used.

$$y_t = \alpha + e_t \tag{A.5}$$

Let $\hat{e}_t$ be the residuals from this regression. Then, it can be shown that

$$\hat{e}_t = e_j - X_j'(X'X)^{-1}X'e + Z_t'\gamma - X_j'(X'X)^{-1}X'Z\gamma$$

where $Z = (Z_1, .., Z_T)'$. Then, we have

$$\frac{1}{\sqrt{T}}\hat{S}_{[rT]} = \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}[e_j - X_j'(X'X)^{-1}X'e] + \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}Z_t'\gamma - \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}X_j'(X'X)^{-1}X'Z\gamma \tag{A.6}$$

The first term follows: $\sigma[W(r) - rW(1)]$, which is shown in KPSS (1992). The second term follows, from Proposition 1.c,

$$\sqrt{T}\frac{1}{T}\sum_{j=1}^{[rT]}[\gamma_1\sin(2\pi kj/T) + \gamma_2\cos(2\pi kj/T)] = \sqrt{T}\,\frac{r}{2\pi k}\left(\gamma_1\left(1-\cos(2\pi kr)\right) + \gamma_2\sin(2\pi kr)\right)$$

$$= O_p(\sqrt{T})$$

The third term in (A.6) is shown to follow

$$-\sqrt{T}\,r\frac{1}{T}\sum_{j=1}^{T}(\gamma_1\sin(2\pi kt/T) + \gamma_2\cos(2\pi kt/T)) = -\sqrt{T}\,r(\gamma_1 s_0 + \gamma_2 c_0) = O_p(\sqrt{T})$$

The third term disappears for integer frequencies k, but the second remains $O_p(\sqrt{T})$. Thus, $\frac{1}{\sqrt{T}}\hat{S}_{[rT]} = O_p(\sqrt{T})$. Then, the numerator of the KPSS is shown to diverge

$$\frac{1}{T^2}\sum_{j=1}^{T}\hat{S}_t^2 = O_p(T)$$

Therefore, the KPSS level statistic ignoring existing unattended nonlinearity will also diverge as the sample size increases.

Next, we examine the case of the trend stationarity test. The DGP implies equation (1) with $X_t = [1, t]$, and $Z_t = [\sin(2\pi kt/T), \cos(2\pi kt/T)]'$ such that a Fourier nonlinear term is present, but Z_t is ignored in the testing regressions where the usual KPSS regression is used.

$$y_t = \alpha + \xi t + e_t \tag{A.7}$$

Let $\hat{e}_t$ be the residuals from this regression. Then, we have

$$\frac{1}{\sqrt{T}}\hat{S}_{[rT]} = \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]}[e_j - X_j'(X'X)^{-1}X'e] + \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]} Z_t'\gamma - \frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]} X_j'R(RX'XR)^{-1}RX'Z\gamma \quad (A.8)$$

where the scaling matrix is given as $R = diag[1/\sqrt{T},\ 1/T^{1.5}]$. The first term approaches the second level standard Brownian bridge, which is shown in KPSS (1992). The second term is $O_p(\sqrt{T})$, as in the case with the level stationary test. For the third term in (A.8), we note

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{[rT]} X_j'R \rightarrow [r,\ \frac{1}{2}r^2], \text{ and } (RX'XR)^{-1} \rightarrow \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}.$$

And, we divide the last product term by $\sqrt{T}$ to obtain

$$\frac{1}{\sqrt{T}}RX'Z\gamma = \begin{pmatrix} \frac{1}{T}\sum_{t=1}^{T}(\gamma_1 \sin(2\pi kt/T) + \gamma_2 \cos(2\pi kt/T)) \\ \frac{1}{T^2}\sum_{t=1}^{T}(\gamma_1 t\cdot \sin(2\pi kt/T) + \gamma_2 \cos(2\pi kt/T)) \end{pmatrix} \rightarrow \begin{bmatrix} (\gamma_1 s_0 + \gamma_2 c_0) \\ (\gamma_1 s_1 + \gamma_2 c_1) \end{bmatrix}$$

Therefore, the third term in (A.8) is $O_p(\sqrt{T})$. Thus, $\frac{1}{\sqrt{T}}\hat{S}_{[rT]} = O_p(\sqrt{T})$. Then, the numerator of the KPSS diverges, and accordingly, the KPSS trend statistic ignoring existing unattended nonlinearity will also diverge as the sample size increases.

Bootstrapping the significance of the F-test

Testing the significance of the trigonometric terms in the presence of stationary (but not necessarily *i.i.d.*) errors is similar to the problem posed in Diebold and Chen (1996). Specifically, Diebold and Chen test the significance of dummy variables added to the Andrews and Ploberger (1994) test for structural break. The linear DGP under the null hypothesis of stationary (but not necessarily *i.i.d.*) errors is:

$$y_t = X_t'\beta + Z_t'\gamma + r_t + \varepsilon_t$$

$$r_t = r_{t-1} + u_t$$

where $\gamma = (0, 0)'$, $\sigma_u^2 = 0$ and $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ with $\rho < 1$. It should be clear that this DGP has the AR(1) representation:

$$y_t = \alpha + \rho\, y_{t-1} + v_t$$

The recursive bootstrap uses the estimated parameters $\hat{\alpha}$ and $\hat{\rho}$ to generate recursive bootstrap realizations of $\{y_t\}$ according to

$$y_t^* = \hat{\alpha} + \hat{\rho}\, y_{t-1}^* + v_t^*.$$

If the null hypothesis is that of trend-stationarity, then this equation will include a trend variable. The bootstrap residuals are random redraws (with replacement) from the estimated AR(1) model for y_t. A starting value for y_0^* is either drawn from the estimated unconditional distribution for y, or alternatively $y_0^* = y_1$ and a number of start-up values are discarded to alleviate the impact of a constant starting value. The recursive bootstrap was shown to deliver consistent inference on the AR parameters by Bose (1988).

The remainder of the procedure proceeds as usual in a bootstrap testing environment. B bootstrap replications for $\{y_t\}$ are used to generate, $\{ y_{t,b}^* \}$ with $b = 1, \ldots, B$. The test statistic $F_i(\hat{k})$, $i = \mu, \tau$, is then calculated for each bootstrap realisation, $F_{i,b}(\hat{k})$. An estimate of the test's p-value for a right-tail test is provided by the proportion of bootstrap test statistics, exceeding $F_i(\hat{k})$ calculated from the observed data.

The results in Table 6 illustrate the feasibility and accuracy of the bootstrap testing strategy at varying levels of residual persistence and sample sizes. Additional details are available from the authors on request.